Theory of the Motion of the Moon; containing a New Calculation of the Expressions for the Coordinates of the Moon in terms of the Time. By Ernest W. Brown, M.A., Sc.D.

PART I. CHAPTERS I.–IV.

INTRODUCTION.

The formation of numerical expressions, deduced as a consequence of the Newtonian laws of motion and gravitation, which shall represent the position of the Moon at any time, may be roughly divided into three stages. As a first step we consider each of the three bodies—the Sun, the Earth, and the Moon—as a sphere of mass equal to its actual mass and arranged in concentric spherical layers of equal density. The Earth (or the centre of mass of the Earth and Moon) is supposed to move round the Sun in a certain ideal elliptic orbit, and all disturbances of this orbit and of the Moon from any other source than the ideal Sun and Earth are neglected. This first stage constitutes nearly the whole of the labour of solving the problem of three bodies as far as the particular configuration of the Sun-Earth-Moon system is concerned. When this is done we proceed to the second step, which involves the determination of the effects due to the difference between the actual and ideal motions of the Earth and Sun, to the influences exerted by the other bodies of the solar system, and to the differences between the real and ideal arrangements of the masses of the bodies. The calculations so far may, theoretically at least, be made without any knowledge of the configuration of the system at any given

time or times beyond a general idea of the order of magnitude of certain of the constants involved. The third and final stage consists in a determination by observation of the various constants which have entered into the theory, and the substitution of their values so as to obtain numerical expressions for the coordinates in terms of the time.

In actual practice these lines of division cannot be satisfactorily kept, partly from the length and complexity of the calculations necessary to obtain algebraical expressions sufficiently complete, and partly from a similar difficulty in comparing large numbers of observations with the results of the theory for the purpose of obtaining the values of the constants involved. Certain of the latter, however, particularly those most frequently occurring, namely, the mean motions of the Sun (or Earth) and Moon, can be obtained, with very little knowledge of the theory, with sufficient accuracy to enable us to use their numerical values at the outset, and so to save a large part of the labour. The first and second steps may be separated without much difficulty, and it is the first which forms the chief object of this Memoir.

Of the methods which have been devised to solve the first part of the problem, those of Hansen and Delaunay must take the first place, not because they were intrinsically the best adapted to the purpose, but because in the hands of their authors they were actually carried out with a degree of detail greater than any other. It is not necessary to discuss here their respective merits; it is sufficient to recall the fact that Hansen's was entirely numerical, and was made the basis of the tables of the Moon used at the present time, although the latter were published forty years ago, while Delaunay's was, owing to the method used, entirely algebraical. Further, Hansen embraced the whole problem in his theory; Delaunay only lived long enough to complete the first step; his work, however, has been carried on by other writers, and will attain its completion in the tables now in process of formation. The earlier method of Laplace, adopted also by Damoiseau and Plana, and that of Lubbock and de Pontécoulant, although perhaps unsuitable to obtain the accuracy required at the present day, attained results which were at the time as much in advance of those previously published as those of Hansen and Delaunay were over all earlier theories.

But if we look only to suggestions of methods of treatment, quite apart from the extent to which they were actually carried out, the most fruitful

contributions to the first part of the problem were undoubtedly those of EULER, who preceded all the writers just mentioned. His two treatises of 1753 and 1772 contain three distinct methods; and it is not a little remarkable that the theories of HANSEN and DELAUNAY may be said to be ultimately based on two of them, while the third forms the foundation of the method developed below. Amongst the points of correspondence between the first theory of EULER and the theory of HANSEN, it is sufficient to note the manner in which the true longitude and the radius vector of the Moon are expressed. HANSEN, however, covered up most of the traces of any such connection by his peculiar method of using a so-called "variable time" and by developing his formulæ with the aid of the method of variation of arbitrary constants. His formulæ can be otherwise obtained, as HANSEN himself and others have shown; even if it were not so, we owe the application of the arbitrary constants method to celestial mechanics in the first place to EULER. This latter method, which is contained in the appendix to EULER's volume of 1753, was rendered a practical one under the masterly treatment of DELAUNAY; but it is doubtful if anyone but the originator of the method would have had the courage to undertake the laborious calculations necessary to bring the work to a successful conclusion.

The theory of EULER, published in 1772, is of particular interest here, since it suggested the method used below. It is based on the use of rectangular axes, of which two move in their own plane with constant angular velocity, and on a division of the inequalities into classes according to the composition of their coefficients. This work of EULER seems to have received but little attention, apart from the practical results obtained; no attempt was made to develop his method further. It was reserved for Dr. G. W. HILL, over a hundred years later, to take the next step by so altering the forms of the equations, while preserving the original ideas on which they were constructed, that they might be made available for accurate calculation. His two papers, "Researches in the Lunar Theory," published in the first volume of the *American Journal of Mathematics*, and "On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon," published separately, and also in the eighth volume of the *Acta Mathematica*, showed what the method was capable of effecting, and opened out a new region for practical calculations and theoretical researches.

"Dans cette œuvre," says M. POINCARÉ in the preface to the first volume of his *Mécanique Céleste*, "il est permis d'apercevoir le germe de la plupart des progrès que la science a faits depuis."

In this connection the work of ADAMS must be mentioned. Starting from an entirely different point of view, that of effecting an accurate determination of the mean motions of the principal arguments, he obtained the corresponding part of the motion of the node by a method somewhat similar to that used by Dr. HILL for the perigee. One of the main difficulties in the latter case—the reduction of the equations to a suitable form—does not occur in the case of the node, and there is no indication in the single paper which ADAMS published on that subject in the *Monthly Notices* for 1877 that he contemplated the use of moving rectangular coordinates. But that he was investigating the properties of the latter may be inferred from another paper on the lunar theory in the same volume of the *Monthly Notices*.

During the last six years I have been attempting to develop the ideas contained in the "Researches" by calculating the coefficients of terms with certain definite characteristics.* Dr. HILL had obtained those which had the characteristic unity, that is, which were functions of the mean motions of the Sun and Moon only, and also that part of the motion of the perigee which was a function of the same quantities; ADAMS had done the same thing for the motion of the node. It therefore remained to obtain the general equations, to put them into forms suitable for calculation, and to show how the other parts of the motions of the perigee and node might be obtained. Experiments were made with the inequalities whose characteristics are the first, second, and third powers of the ratio of the mean parallaxes of the Sun and Moon, and the same powers of the eccentricity of the Moon. The forms of the equations which were there used were, however, troublesome, chiefly from their liability to produce errors in the actual calculations. In a paper which will be hereafter referred to, for the sake of brevity, as the "Investigations,"† I showed how this difficulty might be avoided without, I

* The "characteristic" of any part of a coefficient is defined to be that part in its expression which consists of powers and products of the eccentricities, the inclination, and the ratio of the mean parallaxes. The other factor is a function of the ratio of the mean motions, and it also depends on the nature of the coordinate used. See Chap. I. § 15, below.

† "Investigations in the Lunar Theory," *Amer. Jour. Math.* vol. xvii. pp. 318–358.

think, causing any increase in the labour of making the calculations, and it has certainly diminished the actual time required for their performance. In fact, at least three fourths of the calculations might be performed by a computer whose stock-in-trade amounted to little more than a thorough knowledge of logarithms. An effective control over the computations can be kept at almost every step; and as the operations which would be turned over to the computer are always the same, he would soon be able to do his work with very little supervision.

It is intended to apply the method so as to completely solve the problem of the Moon's motion as far as it is affected by the Sun and Earth alone, the action of the Moon only on these bodies being included, and the three bodies being treated as particles of equal masses. The degree of accuracy aimed at is that the coefficients of all periodic terms in longitude, latitude, and parallax shall be included which are greater than $0''\cdot01$, and that they shall be correct to this amount. The number of terms required is undoubtedly very great. The calculation of coefficients up to the sixth order inclusive with respect to e, k—the lunar eccentricity and inclination—will be necessary; those of the seventh order may be replaced by their elliptic values. The corresponding orders for e', a—the solar eccentricity and the ratio of the mean parallaxes—are deduced from the fact that e'^2 is roughly of the order e^3 or k^3, and that a is of the order e^2 or k^2.

To obtain the coefficients with the above-mentioned degree of accuracy it will be necessary to calculate terms contained in about one hundred characteristics. These will include about five hundred periodic terms, and will require the actual calculation of perhaps two thousand separate coefficients. The results now published contain the terms present in fourteen characteristics—that is to say, about one-seventh of the whole. Notwithstanding the fact that the number of terms contained in the higher characteristics is much greater than that in the lower ones, the work done so far probably amounts to more than one-fifth of the whole. This is due to the fact that a much higher degree of accuracy is required for the lower terms than is actually necessary to obtain the corresponding coefficients correctly to $0''\cdot01$; the presence of small divisors causes a loss of accuracy, which has to be continually borne in mind in judging of the number of places of decimals which are to be calculated. For example, the term, the mean motion of

whose argument is twice the difference between the mean motions of the perigee and node, requires that the calculations be actually carried three places of decimals further than would be necessary for a term of the same order with no small divisor. Fortunately, however, the majority of the terms which cause the most trouble, due to the presence of small divisors, are those which contain both e', a in their characteristics, and therefore the number of characteristics of this nature to be considered is much smaller than would otherwise be the case.

The theory will be an algebraical one throughout, with the single and important exception that the numerical value of the ratio of the mean motions of the Sun and Moon is substituted. The reasons for this may be briefly stated as follows :—First, slow convergence of the series which represent the coefficients arranged according to powers of m or m,* e, k, e', a, takes place mainly along powers of m ; secondly, the value of m is known from observation with great accuracy ; thirdly, estimates would have to be made of the values of the constants e, k used in this theory from the values of differently defined constants of eccentricity and latitude used in other theories ; fourthly, very little, if any, extra trouble is caused by leaving e, k, e', a arbitrary. Thus the theory, while remaining to a large extent algebraical, will possess all the advantages of a purely numerical theory. It may be also mentioned that, by combining the results of this theory with Dr. HILL's modification of DELAUNAY, it can be effectively used for researches by the method of the variation of arbitrary constants. The procedure is intrinsically contained in my paper "On the Theoretical Values of the Secular Acceleration in the Lunar Theory" in the *Monthly Notices* for March 1897.

Of the four chapters which are now published, the first contains the whole theory, with certain exceptions, so far as it is necessary for the continuous development of the numerical results. The exceptions are those parts of the theory which refer to numerical results previously obtained and which are not necessary for those which follow. For example, all details of purely theoretical interest are omitted, and no account is given of Dr. HILL's method for the determination of the intermediate orbit used here (which I

* As usual $m = n'/n$, $\mathrm{m} = n'/(n-n')$, where n, n' are the observed mean motions of the Moon and Sun.

have called the "variation curve"), or of those methods used by him and ADAMS for finding the principal parts of the motions of the perigee and node. All that is necessary is a quotation of the numerical results, and they will be found in their proper places with the sources from which they have been obtained. The following is the table of contents :—

With regard to the calculations, no trouble has been spared to secure correctness. Errors are of two kinds, those which are merely numerical and those which are partly algebraical—*i.e.* due to the use of a wrong series of factors or to the omission of some series of terms. To test the former, equations of verification were computed at every step, and the nature of the method rendered these very numerous. An error of the latter kind may

* The word "order" here and elsewhere refers only to e, e', k, α, and not to m.

escape such a verification, and will generally produce a *large* discordance ; the results were therefore tested by a rough comparison with those of another theory—say, that of Delaunay. One searching test of this kind has been applied to the majority of the terms now published, namely, the comparison of the motions of the perigee and node deduced therefrom with those deduced by Hansen and Delaunay.

The results obtained so far point to certain appreciable errors in the theories of Hansen and Delaunay. Dr. Hill has shown * that the last two terms of Delaunay's expression for the part of the motion of the perigee which depends on m only are wrong. The part of the motion of the perigee which depends on e^2 I have calculated in two quite different ways, and it appears that the last two, if not the last three, of these terms of Delaunay's expression are seriously erroneous.† An error in Delaunay's expression for the part of the motion of the node which depends on e'^2 was actually traced down to an error of transcription in his theory.‡ Hansen's theoretical value for the annual motion of the node appears to be at least one if not two seconds in error.§ Professor Newcomb in his discussion of the results obtained for the coefficient of parallactic inequality || considers that Hansen's value is about $0''\cdot30$ in error : this amount, though small in itself, is of importance if the coefficient be used to obtain the solar parallax. I hope before long to finish the computation of this coefficient so as to obtain it within $0''\cdot02$, and, at a later period, the values of the annual motions of the perigee and node, so far as these depend on solar action only, correct to $0''\cdot01$.

CHAPTER I

General Development of the Theory.

Section (i).—*The Problem of Three Bodies and the Disturbing Function.*

1. The disturbing function used by all investigators except Hansen gives only a portion of the inequalities produced by the Sun in the motion of the Moon. This form of the disturbing function will be used below, and

* *Annals of Mathematics*, vol. ix. pp. 31–41.

† *Monthly Notices*, vol. lvii. p. 336.

‡ *Ibid.* p. 341.

§ *Ibid.* p. 340.

|| *Astron. Jour.* vol. xv. p. 167.

therefore we shall, in this section, investigate the small additions which must be made to it in order that the whole of the effect of the Sun on the Moon's motion may be obtained. The method of this section is similar to that of Prof. NEWCOMB in "The Actions of the Planets on the Moon" (*Amer. Eph. Papers*, vol. v).

Let X, Y, z, r be the coordinates and distance of the Moon referred to axes fixed in direction and passing through the centre of the Earth; x', y', z', r', those of the Sun referred to parallel axes through the centre of mass of the Earth and Moon; M, E, m', the masses of the Moon, Earth, and Sun reckoned in astronomical units. The x-coordinates of the Moon, Earth, and Sun referred to parallel axes through the centre of mass of the three bodies are respectively

$$\frac{EX}{E+M}-\frac{m'\,x'}{m'+E+M}, \qquad -\frac{MX}{E+M}-\frac{m'\,x'}{m'+E+M}, \qquad \frac{(E+M)\,x'}{m'+E+M},$$

with similar expressions for the other coordinates.

Let T be the kinetic energy of the system relative to the centre of mass. Then

$$\begin{aligned} 2T = {} & M\left\{\left(\frac{E}{E+M}\frac{dX}{dt}-\frac{m'}{m'+E+M}\frac{dx'}{dt}\right)^2+\ldots+\ldots\right\} \\ & +E\left\{\left(\frac{M}{E+M}\frac{dX}{dt}+\frac{m'}{m'+E+M}\frac{dx'}{dt}\right)^2+\ldots+\ldots\right\} \\ & +m'\left\{\left(\frac{E+M}{m'+E+M}\frac{dx'}{dt}\right)^2+\ldots+\ldots\right\} \\ = {} & \mu_1\,(\dot{X}^2+\dot{Y}^2+\dot{z}^2)+\mu_2\,(\dot{x}'^2+\dot{y}'^2+\dot{z}'^2), \end{aligned}$$

where

$$\mu_1=\frac{EM}{E+M}, \qquad \mu_2=\frac{m'\,(E+M)}{m'+E+M}.$$

Let F be the potential energy of the system. By LAGRANGE's equations we then have for the Moon's motion relative to the Earth—

$$\mu_1\frac{d^2X}{dt^2}=\frac{\partial F}{\partial X}, \qquad \mu_1\frac{d^2Y}{dt^2}=\frac{\partial F}{\partial Y}, \qquad \mu_1\frac{d^2z}{dt^2}=\frac{\partial F}{\partial z},$$

and for the Sun's motion relative to the centre of mass of the Earth and Moon—

$$\mu_2\frac{d^2x'}{dt^2}=\frac{\partial F}{\partial x'}, \qquad \mu_2\frac{d^2y'}{dt^2}=\frac{\partial F}{\partial y'}, \qquad \mu_2\frac{\partial^2 z'}{dt^2}=\frac{\partial F}{\partial z'},$$

F being expressed in terms of X, Y, z, x', y', z'.

Let r_1' be the distance of the Sun from the Earth, Δ its distance from the Moon. Then

$$F=\frac{EM}{r}+\frac{Em'}{r_1'}+\frac{Mm'}{\Delta};$$

and

$$r^2=X^2+Y^2+z^2, \qquad r'^2=x'^2+y'^2+z'^2,$$

$$r_1'^2=r'^2+2\frac{M}{E+M}r'r\mathrm{S}+\frac{M^2}{(E+M)^2}r^2,$$

$$\Delta^2=r'^2-2\frac{E}{E+M}r'r\mathrm{S}+\frac{E^2}{(E+M)^2}r^2,$$

where

$$rr'\mathrm{S}=Xx'+Yy'+zz'.$$

S is therefore the cosine of the angle subtended by Δ at the centre of mass of the Earth and Moon.

2. We shall first consider the effect of the Moon's motion on the Sun and determine the deviation of the Sun's motion from an elliptic orbit in the plane of reference. For this purpose we may omit the first term of F, since it does not contain x', y', z'. Then

$$\frac{F}{\mu_2}=\frac{m'+E+M}{E+M}\left(\frac{E}{r_1'}+\frac{M}{\Delta}\right).$$

Expanding $1/r_1'$, $1/\Delta$ in powers of r/r', we obtain

$$\frac{F}{\mu_2}=(m'+E+M)\left[\frac{1}{r'}+\frac{EM}{(E+M)^2}\frac{r^2}{r'^3}\left(\frac{3}{2}\mathrm{S}^2-\frac{1}{2}\right)+\ldots\right].$$

The order of the second term in comparison with the first is in the ratio

$$Mr^2 : Er'^2=1 : 12,000,000$$

approximately. The order of the next term is in the ratio $Mr^3 : Er'^3$, a quantity which may and will be totally neglected. A sufficient correction to the elliptic motion of the Sun about the centre of mass of the Earth and Moon may therefore be obtained by using

$$(m'+E+M)\frac{EM}{(E+M)^2}\frac{r^2}{r'^3}\left(\frac{3}{2}\mathrm{S}^2-\frac{1}{2}\right)$$

as the disturbing function, and substituting for the coordinates of the Moon their elliptic values, together with the principal inequalities due to the Sun,

as found in Chaps. II., III. below. If n' be the observed mean motion of the Sun, we define the mean distance a' by the equation

$$m'+E+M=n'^2a'^3.$$

3. Next, for the motion of the Moon we have, on expanding F and rejecting the useless term $m'(E+M)/r'$,

$$\begin{aligned}\frac{F}{\mu_1}&=\frac{E+M}{r}+\frac{m'(E+M)}{EM}\left(\frac{E}{r_1'}+\frac{M}{\Delta}\right)\\
&=\frac{E+M}{r}+\frac{m'r^2}{r'^3}\left[\left(\frac{3}{2}S^2-\frac{1}{2}\right)+\frac{E-M}{E+M}\frac{r}{r'}\left(\frac{5}{2}S^3-\frac{3}{2}S\right)\right.\\
&\qquad+\frac{E^2-EM+M^2}{(E+M)^2}\frac{r^2}{r'^2}\left(\frac{35}{8}S^4-\frac{15}{4}S^2+\frac{3}{8}\right)\\
&\qquad\left.+\frac{E^3-E^2M+EM^2-M^3}{(E+M)^3}\frac{r^3}{r'^3}\left(\frac{63}{8}S^5-\frac{35}{4}S^3+\frac{15}{8}S\right)+\dots\right].\end{aligned}$$

The expansion is carried as far as will be necessary, since we shall neglect quantities of the orders r^4/r'^4, Mr^3/Er'^3.

The force-function which we shall use is that ordinarily used, namely,

$$\begin{aligned}\Omega&=\frac{E+M}{r}+m'\left[\frac{1}{\sqrt{r'^2-2rr'S+r^2}}-\frac{1}{r'}-\frac{rS}{r'^2}\right]\\
&=\frac{E+M}{r}+n'^2a'^3\frac{r^2}{r'^3}\left[\left(\frac{3}{2}S^2-\frac{1}{2}\right)+\frac{r}{r'}\left(\frac{5}{2}S^3-\frac{3}{2}S\right)\right.\\
&\qquad+\frac{r^2}{r'^2}\left(\frac{35}{8}S^4-\frac{15}{4}S^2+\frac{3}{8}\right)\\
&\qquad\left.+\frac{r^3}{r'^3}\left(\frac{63}{8}S^5-\frac{35}{4}S^3+\frac{15}{8}S\right)+\dots\right] \qquad \dots \qquad (1)\end{aligned}$$

In this, the elliptic values of x', y' ($z'=0$) are substituted and m' is put equal to $n'^2a'^3$ instead of $n'^2a'^3-E-M$. It is necessary to consider what corrections must be made when Ω is used instead of F/μ_1.

4. The corrections are of three kinds and are so small that we need only consider their effects to the first order of the disturbance.

(*a*) Correction due to putting $n'^2a'^3=m'$ instead of $m'+E+M$. We must add to Ω the terms

$$-(E+M)\frac{r^2}{r'^3}\left[\frac{3}{2}S^2-\frac{1}{2}+\frac{r}{r'}\left(\frac{5}{2}S^3-\frac{3}{2}S\right)\right].$$

This correction will be sufficiently accounted for if we multiply all the inequalities due to the Sun by

$$1-\frac{E+M}{m'}=1-\frac{1}{330\,000},$$

approximately.*

(*b*) Correction due to using the elliptic instead of the true values of the solar coordinates. We must add to Ω the term

$$n'^2a'^3\delta\left[\frac{r^2}{r'^3}\left(\frac{3}{2}S^2-\frac{1}{2}\right)\right],$$

where δ operates on x', y', z', and $\delta x'$, $\delta y'$, $\delta z'$ are the perturbations of x', y', z', as found in § 2.

(*c*) Correction due to the difference between F/μ_1 and Ω. Since the terms in F/μ_1 or Ω involving $n'^2a'^3\frac{r^2}{r'^3}\left(\frac{r}{r'}\right)^j$ give inequalities having the factor $(a/a')^j$, where a is the "constant" of the Moon's distance, this correction may be partly made by multiplying the inequalities having the factor $(a/a')^j$ by

$$\left(\frac{E-M}{E+M}\right)^j.$$

To the order considered here, there will then remain to be added to Ω the term

$$n'^2a'^3\frac{r^2}{r'^3}\left[\frac{EM}{(E+M)^2}\cdot\frac{r^2}{r'^2}\left(\frac{35}{8}S^4-\frac{15}{4}S^2+\frac{3}{8}\right)\right].$$

5. The method of procedure is therefore as follows:—The values of X, Y, z are first obtained by using Ω with $m'=n'^2a'^3$, $z'=0$ and elliptic values for x', y'. With the values of X, Y, z thus formed we compute $\delta x'$, $\delta y'$, $\delta z'$ by means of the disturbing function in § 2. The corrections to X, Y, z, noted in (*a*), (*b*), (*c*) above, are then easily obtained. The first step is that

* Except for the mean motion of the perigee where the second and succeeding terms, which are of the order of the square and higher powers of the disturbing forces, are a little greater in actual value than the first term. The correction may be made with sufficient accuracy by multiplying the mean motion of the perigee by

$$1-\frac{1}{2}\frac{E+M}{m'}-\frac{1}{2}\cdot 2\,\frac{E+M}{m'}=1-\frac{3}{2}\frac{E+M}{m'}.$$

See *Monthly Notices*, 1897 June.

which very frequently bears the name of the "Lunar Theory." It will be noticed from this investigation that the elliptic values of the solar coordinates to be used are those referred to the centre of mass of the Earth and Moon.

Section (ii).—*The Equations of Motion.*

6. Let

$x, y, z,$ be the coordinates of the Moon, referred to rectangular axes through the Earth's centre, of which those of x, y are in the plane of the Sun's orbit (supposed fixed and elliptic), the positive direction of the moving x-axis being constantly directed to the mean place of the Sun ;

$r^2=x^2+y^2+z^2$;

$\rho^2=x^2+y^2$;

n, n', the observed mean motions of the Moon and Sun ;

r', e', a', the radius vector, eccentricity, and semi-axis major of the Sun's orbit (§ 5) ;

v, the solar equation of the centre ;

$S_1=x \cos v+y \sin v$.

So that

$$rr'S=Xx'+Yy'+zz'=r'(x \cos v+y \sin v)=r'S_1,$$

and

$$r^2=\rho^2+z^2.$$

Also, representing $\sqrt{-1}$ by ι, let

$$u=x+y\iota, \qquad s=x-y\iota, \qquad us=\rho^2 ;$$

$$\mathrm{m}=\frac{n'}{n-n'}, \qquad \kappa=\frac{E+M}{(n-n')^2} ;$$

$$\zeta=\exp.\,(n-n')(t-t_0)\iota,$$

$$D=\frac{1}{(n-n')\iota}\frac{d}{dt}=\zeta\frac{d}{d\zeta} ;$$

where t_0 is a constant to be defined later.

The equations of motion, referred to the moving axes and with the force-function Ω (§ 3), are

$$\frac{d^2x}{dt^2}-2n'\frac{dy}{dt}-n'^2x=\frac{\partial\Omega}{\partial x},$$

$$\frac{d^2y}{dt^2}+2n'\frac{dy}{dt}-n'^2y=\frac{\partial\Omega}{\partial y},$$

$$\frac{d^2z}{dt^2} \qquad\qquad =\frac{\partial\Omega}{\partial z}.$$

Let

$$\Omega' = \Omega + \frac{1}{2}n'^2(x^2+y^2) = \Omega + \frac{1}{2}n'^2 us,$$

and transform to the independent variables u, s, z and the dependent variable ζ. The equations become

$$D^2u + 2\text{m}Du = -\frac{2}{(n-n')^2}\frac{\partial\Omega'}{\partial s},$$

$$D^2s - 2\text{m}Ds = -\frac{2}{(n-n')^2}\frac{\partial\Omega'}{\partial u},$$

$$D^2z \qquad = -\frac{1}{(n-n')^2}\frac{\partial\Omega'}{\partial z},$$

where, by Sect. (i),

$$\Omega' = \frac{E+M}{(us+z^2)^{\frac{1}{2}}} + n'^2a'^3\left[\frac{1}{(r'^2-2r'\text{S}_1+us+z^2)^{\frac{1}{2}}} - \frac{1}{r'} - \frac{\text{S}_1}{r'^2}\right] + \frac{1}{2}n'^2us.$$

This gives, on expansion according to powers of $1/r'$, after some transformations,

$$\frac{2}{(n-n')^2}\Omega' = \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} + \frac{3}{4}\text{m}^2(u+s)^2 - \text{m}^2z^2 + \Omega_1$$

where

$$\left.\begin{aligned}\Omega_1 = {} & 3\text{m}^2\left[\frac{a'^3}{r'^3}\text{S}_1^2 - \frac{1}{4}(u+s)^2\right] - \text{m}^2(us+z^2)\left(\frac{a'^3}{r'^3} - 1\right) \\ & + \frac{\text{m}^2}{a'}\cdot\frac{a'^4}{r'^4}\left[5\text{S}_1^3 - 3\text{S}_1(us+z^2)\right] \\ & + \frac{\text{m}^2}{a'^2}\cdot\frac{a'^5}{r'^5}\left[\frac{35}{4}\text{S}_1^4 - \frac{15}{2}\text{S}_1^2(us+z^2) + \frac{3}{4}(us+z^2)^2\right] \\ & + \frac{\text{m}^2}{a'^3}\cdot\frac{a'^6}{r'^6}\left[\frac{63}{4}\text{S}_1^5 - \frac{35}{2}\text{S}_1^3(us+z^2) + \frac{15}{4}\text{S}_1(us+z^2)^2\right] \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

$$= \omega_2 + \omega_3 + \omega_4 + \omega_5 + \ldots ,$$

suppose, where ω_q is the part of Ω_1 which involves u, s, z to the degree q; S_1 is of the first degree with respect to u, s. It will be noticed that ω_2 is zero when $e' = 0$.

The equations may now be written—

$$(D+\mathrm{m})^2u+\frac{1}{2}\mathrm{m}^2u+\frac{3}{2}\mathrm{m}^2s-\frac{\kappa u}{(us+z^2)^{\frac{3}{2}}}=-\frac{\partial\Omega_1}{\partial s}, \qquad \ldots \qquad \ldots \qquad (3)$$

$$(D-\mathrm{m})^2s+\frac{1}{2}\mathrm{m}^2s+\frac{3}{2}\mathrm{m}^2u-\frac{\kappa s}{(us+z^2)^{\frac{3}{2}}}=-\frac{\partial\Omega_1}{\partial u}, \qquad \ldots \qquad \ldots \qquad (3')$$

$$(D^2-\mathrm{m}^2)z-\frac{\kappa z}{(us+z^2)^{\frac{3}{2}}}=-\frac{1}{2}\frac{\partial\Omega_1}{\partial z} \qquad \ldots \qquad \ldots \qquad (4)$$

These are the fundamental equations in the theory. Since u, s are conjugate complexes, either of the first two equations is sufficient; we shall use the first in the calculations.

7. *Homogeneous Form of the Equations.*—Multiply the three equations by Ds, Du, $2Dz$ respectively, and add. We obtain

$$D\left[Du\,.\,Ds+(Dz)^2+\frac{3}{4}\mathrm{m}^2(u+s)^2-\mathrm{m}^2z^2+\frac{2\kappa}{(us+z^2)^{\frac{1}{2}}}\right]$$
$$=-\left[\frac{\partial\Omega_1}{\partial s}Ds+\frac{\partial\Omega_1}{\partial u}Du+\frac{\partial\Omega_1}{\partial z}Dz\right].$$

Since Ω_1 is a function of u, s, z, r', $\mathbf{v}$ only, and since r', $\mathbf{v}$ are supposed to be known functions of the time,

$$D\Omega_1=\frac{\partial\Omega_1}{\partial u}Du+\frac{\partial\Omega_1}{\partial s}Ds+\frac{\partial\Omega_1}{\partial z}Dz+\frac{\partial\Omega_1}{\partial t}Dt\,;$$

and therefore the right-hand member of the previous equation

$$=\frac{\partial\Omega_1}{\partial t}Dt-D\Omega_1=\frac{1}{(n-n')\iota}\frac{\partial\Omega_1}{\partial t}-D\Omega_1$$
$$=D'\Omega_1-D\Omega_1=D[D^{-1}(D'\Omega_1)-\Omega_1].$$

where D^{-1} denotes the operation inverse to D (*i.e.* integration with respect to ζ followed by a division by ζ), and $D'\Omega_1$ denotes the operation D performed on Ω_1 only so far as ζ occurs in r', $\mathbf{v}$.

With this form of the right-hand member we can integrate and obtain

$$Du\,.\,Ds+(Dz)^2+\frac{3}{4}\mathrm{m}^2(u+s)^2-\mathrm{m}^2z^2+\frac{2\kappa}{(us+z^2)^{\frac{1}{2}}}=C'-\Omega_1+D^{-1}(D'\Omega_1) \qquad (5)$$

Now add this to the sum of (3), (3′), (4) multiplied by s, u, $2z$ respectively. Since, by Euler's theorem,

$$s\frac{\partial\Omega_1}{\partial s}+u\frac{\partial\Omega_1}{\partial u}+z\frac{\partial\Omega_1}{\partial z}=\sum_{q=2}^{\infty}q\omega_q$$

we obtain an equation which may be written—

$$D^2(us+z^2)-Du\,.\,Ds-(Dz)^2-2\mathrm{m}(uDs-sDu)+\frac{9}{4}\mathrm{m}^2(u+s)^2-3\mathrm{m}^2z^2$$
$$=C'-\sum_{q=2}^{\infty}(q+1)\omega_q+D^{-1}(D'\Omega_1) \quad \ldots \quad \ldots \quad \ldots \qquad (6)$$

Also multiply (3) by s, (3′) by u and subtract; multiply (3) by z and (4) by u and subtract. The two resulting equations are

$$D(uDs-sDu-2\mathrm{m}us)+\frac{3}{2}\mathrm{m}^2(u^2-s^2)=s\frac{\partial\Omega_1}{\partial s}-u\frac{\partial\Omega_1}{\partial u} \quad \ldots \quad \ldots \qquad (7)$$

$$D(uDz-zDu)-2\mathrm{m}zDu-\mathrm{m}^2uz-\frac{3}{2}\mathrm{m}^2z(u+s)=z\frac{\partial\Omega_1}{\partial s}-\frac{1}{2}u\frac{\partial\Omega_1}{\partial z} \quad \ldots \qquad (8)$$

Instead of the last we may obtain the more symmetrical form,

$$D[(u-s)Dz-zD(u-s)]-2\mathrm{m}zD(u+s)-\mathrm{m}^2z(u-s)$$
$$=z\left(\frac{\partial\Omega_1}{\partial s}-\frac{\partial\Omega_1}{\partial u}\right)-\frac{1}{2}(u-s)\frac{\partial\Omega_1}{\partial z} \qquad (8')$$

The equations (6), (7), (8) will be called the *homogeneous equations*. Their left-hand members are homogeneous and of the second degree with respect to u, s, z, while their right-hand members are, abstraction being made of C', of the 2nd, 3rd . . . degrees with respect to the same variables.*

Section (iii.).—*Development of* Ω_1.

8. In the last section, the development of Ω_1 according to powers of $1/a'$ has been given; the result is there numbered (2). We must now further develop it according to powers of e' and z, which are small quantities of the first order. The development will not be carried beyond quantities of the orders—

$$\frac{\mathrm{a}^3}{a'^3},\quad \frac{\mathrm{a}^2}{a'^2}e',\quad \frac{\mathrm{a}^2}{a'^2}z^2,\quad \frac{\mathrm{a}}{a'}e'^3,\quad e'^5.$$

By the definition of S_1 we have

$$\mathrm{S}_1=x\cos v+y\sin v$$
$$=\frac{1}{2}(ue^{-v\iota}+se^{v\iota})$$

* Fuller explanations of the transformations in the section are given in the *Treatise on the Lunar Theory*, Chap. II. (iii.).

Substituting in (2) and remembering that

$$\Omega_1=\omega_2+\omega_3+\omega_4+\omega_5,$$

we have, to the orders of small quantities just mentioned,

$$\omega_2=\mathrm{m}^2\left[\frac{3}{4}(u^2\mathfrak{a}_2+s^2\bar{\mathfrak{a}}_2)+\frac{1}{2}us\mathfrak{b}_2-z^2\mathfrak{b}_2\right],$$

$$\omega_3=\frac{\mathrm{m}^2}{a'}\left[\frac{5}{8}(u^3\mathfrak{a}_3+s^3\bar{\mathfrak{a}}_3)+\frac{3}{8}(u^2s\mathfrak{c}_3+us^2\bar{\mathfrak{c}}_3)-\frac{3}{2}uz^2\mathfrak{c}_3-\frac{3}{2}sz^2\bar{\mathfrak{c}}_3\right],$$

$$\omega_4=\frac{\mathrm{m}^2}{a'^2}\left[\frac{35}{64}(u^4\mathfrak{a}_4+s^4\bar{\mathfrak{a}}_4)+\frac{5}{16}(u^3s\mathfrak{c}_4+us^3\bar{\mathfrak{c}}_4)+\frac{9}{32}u^2s^2\mathfrak{b}_4\right.$$
$$\left.-z^2\left(\frac{15}{8}u^2\mathfrak{c}_4+\frac{15}{8}s^2\bar{\mathfrak{c}}_4+\frac{9}{4}us\mathfrak{b}_4\right)\right],$$

$$\omega_5=\frac{\mathrm{m}^2}{a'^3}\left[\frac{63}{128}(u^5+s^5)+\frac{35}{128}(u^4s+us^4)+\frac{15}{64}(u^3s^2+u^2s^3)\right];$$

where

$$\mathfrak{a}_2=\frac{a'^3}{r'^3}e^{-2v\iota}-1,\qquad \mathfrak{a}_3=\frac{a'^4}{r'^4}e^{-3v\iota},\qquad \mathfrak{a}_4=\frac{a'^5}{r'^5}e^{-4v\iota},$$

$$\mathfrak{b}_2=\frac{a'^3}{r'^3}-1,\qquad \mathfrak{b}_4=\frac{a'^5}{r'^5},$$

$$\mathfrak{c}_3=\frac{a'^4}{r'^4}e^{-v\iota},\qquad \mathfrak{c}_4=\frac{a'^5}{r'^5}e^{-2v\iota};$$

and $\bar{\mathfrak{a}}_2$, $\bar{\mathfrak{b}}_2$, . . are the values of $\mathfrak{a}_2$, $\mathfrak{b}_2$, . . when $-\iota$ has been put for ι.

9. The quantities $\mathfrak{a}_2$, $\mathfrak{b}_2$, ... are to be expanded in powers of e'; they are well-known elliptic expansions and they have been computed by several investigators. They may be conveniently obtained from the expressions given by Delaunay in chap. ii. vol. i. of his *Théorie de la Lune* by giving to the angle, there called α, suitable values, or from the tables of Cayley (*Mem. R.A.S.* vol. xxix., *Coll. Works*, vol. iii).

Putting $\qquad \text{Exp. } l'\iota=\zeta^{\mathrm{m}} \qquad$ (see § 11, below),

where l' is the solar mean anomaly, we obtain

$$\begin{aligned}\mathfrak{a}_2= & -\frac{5}{2}e'^2+\frac{13}{16}e'^4\\
& +\left(-\frac{1}{2}e'+\frac{1}{16}e'^3-\frac{5}{384}e'^5\right)\zeta^{\mathrm{m}}+\left(\frac{7}{2}e'-\frac{123}{16}e'^3+\frac{489}{128}e'^5\right)\zeta^{-\mathrm{m}}\\
& +0\,.\,\zeta^{2\mathrm{m}} \qquad\qquad\qquad\qquad\qquad +\left(\frac{17}{2}e'^2-\frac{115}{6}e'^4\right)\zeta^{-2\mathrm{m}}\\
& +\left(\frac{1}{48}e'^3+\frac{11}{768}e'^5\right)\zeta^{3\mathrm{m}} \qquad\quad +\left(\frac{845}{48}e'^3-\frac{32525}{768}e'^5\right)\zeta^{-3\mathrm{m}}\\
& +\frac{1}{24}e'^4\zeta^{4\mathrm{m}} \qquad\qquad\qquad\qquad +\frac{533}{16}e'^4\zeta^{-4\mathrm{m}}\\
& +\frac{81}{1280}e'^5\zeta^{5\mathrm{m}} \qquad\qquad\qquad\quad +\frac{228347}{3840}e'^5\zeta^{-5\mathrm{m}}\end{aligned}$$

$$\mathfrak{b}_2=\frac{3}{2}e'^2+\frac{15}{8}e'^4+\left(\frac{3}{2}e'+\frac{27}{16}e'^3+\frac{261}{128}e'^5\right)\left(\zeta^{\mathrm{m}}+\zeta^{-\mathrm{m}}\right)$$
$$+\left(\frac{9}{4}e'^2+\frac{7}{4}e'^4\right)\left(\zeta^{2\mathrm{m}}+\zeta^{-2\mathrm{m}}\right)$$
$$+\left(\frac{53}{16}e'^3+\frac{393}{256}e'^5\right)\left(\zeta^{3\mathrm{m}}+\zeta^{-3\mathrm{m}}\right)$$
$$+\frac{77}{16}e'^4\left(\zeta^{4\mathrm{m}}+\zeta^{-4\mathrm{m}}\right)$$
$$+\frac{1773}{256}e'^5\left(\zeta^{5\mathrm{m}}+\zeta^{-5\mathrm{m}}\right),$$

$$\mathfrak{a}_3=1-6e'^2+\left(-e'+\frac{5}{4}e'^3\right)\zeta^{\mathrm{m}}+\left(5e'-22e'^3\right)\zeta^{-\mathrm{m}}$$
$$+\frac{1}{8}e'^2\zeta^{2\mathrm{m}}\qquad+\frac{127}{8}e'^2\zeta^{-2\mathrm{m}}$$
$$+0\,.\,e'^3\zeta^{3\mathrm{m}}\qquad+\frac{163}{4}e'^3\zeta^{-3\mathrm{m}},$$

$$\mathfrak{c}_3=1+2e'^2+\left(e'+\frac{5}{2}e'^3\right)\zeta^{\mathrm{m}}+\left(3e'+\frac{11}{4}e'^3\right)\zeta^{-\mathrm{m}}$$
$$+\frac{11}{8}e'^2\zeta^{2\mathrm{m}}\qquad+\frac{53}{8}e'^2\zeta^{-2\mathrm{m}}$$
$$+\frac{23}{12}e'^3\zeta^{3\mathrm{m}}\qquad+\frac{77}{6}e'^3\zeta^{-3\mathrm{m}},$$

$$\mathfrak{a}_4=1-\frac{3}{2}e'\zeta^{\mathrm{m}}+\frac{13}{2}e'\zeta^{-\mathrm{m}},$$

$$\mathfrak{c}_4=1+\frac{1}{2}e'\zeta^{\mathrm{m}}+\frac{9}{2}e'\zeta^{-\mathrm{m}},$$

$$\mathfrak{b}_4=1+\frac{5}{2}e'\left(\zeta^{\mathrm{m}}+\zeta^{-\mathrm{m}}\right).$$

The values of $\bar{\mathfrak{a}}_2$, $\bar{\mathfrak{c}}_2$, . . . are obtained from those of $\mathfrak{a}_2$, $\mathfrak{c}_2$, . . . by putting $1/\zeta$ for ζ.

Section (iv).—*Form of the Solution. Notation.*

10. Let V be the true longitude in the plane of $X\,Y$ or $x\,y$ reckoned from the fixed axis of X and ϕ the latitude above this plane. Also let D, l, l', F denote the same angles as in Delaunay's theory, Chap. xi. vol. ii. *i.e.* let

D $=(n-n')\,t+\epsilon-\epsilon'=$ Half arg. of the "Variation,"
$l\ =cnt+\epsilon-\varpi\qquad =$ Arg. of the Principal Elliptic Term,
$l'=n't+\epsilon'-\varpi'\qquad =$,, ,, "Annual Equation,"
F $=gnt+\epsilon-\theta\qquad =$,, ,, Principal term in Latitude.

Here, as usual, ϵ, ϖ, θ are the values of the mean longitudes of the Moon and of its perigee and node at time $t=0$; ϵ' is the mean longitude of the Sun at time $t=0$ and ϖ' the (constant) longitude of its perigee; $(1-c)n$, $(1-g)n$ are the mean motions of the perigee and node.

We then have

$$X=\rho \cos V, \quad Y=\rho \sin V, \quad z=\rho \tan \phi=r \sin \phi,$$

Whence, as we shall put $\zeta = \exp. \mathrm{D}\iota$, (§ 11),

$$x=\rho \cos (V-n't-\epsilon')=\rho \cos (V-nt-\epsilon+\mathrm{D}),$$
$$y=\rho \sin (V-n't-\epsilon')=\rho \sin (V-nt-\epsilon+\mathrm{D});$$

$$\left.\begin{array}{ll} u \;\; =\rho \exp. (V-nt-\epsilon+\mathrm{D})\,\iota & s=\rho \exp. -(V-nt-\epsilon+\mathrm{D})\,\iota \\ u\zeta^{-1}=\rho \exp. (V-nt-\epsilon)\,\iota, & s\zeta=\rho \exp. -(V-nt-\epsilon)\,\iota \end{array}\right\} \quad \ldots \quad \ldots \quad (9)$$

It is well known that, with the limitations here imposed, r, $V-nt-\epsilon$, ϕ are expressible by sums of periodic terms whose arguments are (algebraic) sums of multiples of the four angles D, l, l', F. Hence x, y, z are expressible in the forms

$$\left.\begin{array}{l} x \\ y \\ z \end{array}\right\} = \mathrm{a}\Sigma A_{i,p,q,r} \left.\begin{array}{l} \cos \\ \sin \\ \sin \end{array}\right\} (i\,\mathrm{D}+pl+rl'+q\mathrm{F}), \qquad i, p, q, r=0, \pm 1, \pm 2, \ldots$$

where a is a linear constant and A a coefficient; the sign of summation Σ denotes that the sum of all such terms must be taken. Remembering the definition of u, s, it is evident that u, s, $z\iota$ may be therefore expressed in the form

$$u, s, z\iota=\mathrm{a}\Sigma A_{i,p,q,r} \exp. (i\,\mathrm{D}+pl+rl'+q\mathrm{F})\,\iota,$$

with certain limitations which will be set down later.

There are four sets of notations required. First, for the exponentials; secondly, for the constants of distance, eccentricity, latitude and ratio of the mean parallaxes; thirdly, for the numerical parts of the coefficients (these parts are functions of m only, and the numerical value of m is used throughout); fourthly, for the terms of u, s, z which are of a particular order.

11. *Notation for the Exponentials.*—Recalling that $\mathrm{m}=n'/(n-n')$, we may evidently write

$$\mathrm{D}=(n-n')(t-t_0),$$
$$l=\mathrm{c}\,(n-n')(t-t_1),$$
$$l'=\mathrm{m}\,(n-n')(t-t_3),$$
$$\mathrm{F}=\mathrm{g}\,(n-n')(t-t_2),$$

where

$$\mathrm{c}\,(n-n')=cn, \qquad \mathrm{g}\,(n-n')=gn,$$

and the signification of t_0, t_1, t_2, t_3 is obvious.

We have, in Section (ii), defined ζ by the equation

$$\zeta=\exp.\,(n-n')(t-t_0)\,\iota,$$

and we now give to t_0 in this expression the same meaning as it has in D, so that $\zeta=\exp.\ \mathrm{D}\iota$.

Let, for a moment,

$$\zeta_{\mathrm{c}}^{\mathrm{c}}=\exp.\,\mathrm{c}(n-n')(t-t_1)\iota.$$

Remembering the definition of the operator D we have

$$D^j(\zeta^i\zeta_{\mathrm{c}}^{p\mathrm{c}})=(i+p\mathrm{c})^j\zeta^i\zeta_{\mathrm{c}}^{p\mathrm{c}},$$

where i, j, p are positive or negative integers.

Now, in the method pursued here, we shall always proceed by equating to zero the coefficients of like powers of ζ, $\zeta_{\mathrm{c}}^{\mathrm{c}}$ in equations which consist of such expressions as that just written down, and it will never be necessary to substitute the value of c in the *indices*; its value is only substituted in the coefficients. The above equation shows that the coefficients will be the same whether we write $\zeta^i\zeta_{\mathrm{c}}^{p\mathrm{c}}$, or $\zeta^{i+p\mathrm{c}}$. Further, as the suffix c always occurs in the index whenever it is present as a suffix, the *suffix* is unnecessary for purposes of distinction and we shall omit it in future. The same remarks apply to

$$\zeta_{\mathrm{m}}^{r\mathrm{m}}=\exp.\,rl'\iota, \qquad \zeta_{\mathrm{g}}^{q\mathrm{g}}=\exp.\,q\mathrm{F}\iota.$$

We may therefore put

$$\exp.\,(i\mathrm{D}+pl+rl'+q\mathrm{F})\iota=\zeta^{i+p\mathrm{c}+r\mathrm{m}+q\mathrm{g}},$$

the index of ζ always denoting the coefficient of t in the corresponding argument divided by $n-n'$.

12. *Notation for the Arbitrary Constants and the Parameters.*—There are six arbitrary constants present in the solution. Three of these—ϵ, ϖ, θ, or t_0, t_1, t_2—have already been considered; they are contained in D, l, F, or in ζ, ζ^c, ζ^g. The other arbitraries to be used will be denoted by

$$n, \mathrm{a}, \mathrm{e}, \mathrm{k},$$

connected by one relation. The exact definitions of e, k,—the constants of eccentricity and inclination—will be found in Section (vi); n has been defined as the observed mean motion; the linear constant, a, is connected with n, e, k, $E+M$ by a relation which will be defined in the same Section. In elliptic motion, this relation would be $n^2\mathrm{a}^3=E+M$; in the actual case the relation differs a little from this.

The parameters in powers of which expansion will be made are

$$\mathrm{m}, \mathrm{e}, e', \mathrm{k}, \alpha=\frac{\mathrm{a}}{a'}.$$

The numerical value of m is used, but the other parameters are left arbitrary.

13. *Notation for the Numerical Coefficients.*—From what precedes, it is evident that u, s, $z\iota$ may be expressed in the form

$$u,\ s,\ z\iota=\mathrm{a}\Sigma A_{i,\,p,\,q,\,r}\,\zeta^{i+p\mathrm{c}+r\mathrm{m}+q\mathrm{g}},$$

or, as it will be more convenient to write it,

$$u\zeta^{-1},\ s\zeta,\ z\iota=\mathrm{a}\Sigma A_{i,\,p,\,q,\,r}\,\zeta^{i+p\mathrm{c}+r\mathrm{m}+q\mathrm{g}}, \qquad i, p, q, r=+\infty\ \ldots -\infty.$$

The latter form has the following properties, which are easily deduced from the known properties of the expressions for V, r, ϕ.

(*a*) $s\zeta$ is deduced from $u\zeta^{-1}$ by putting $1/\zeta$ for ζ;
 if $1/\zeta$ be put for ζ in the expression for $z\iota$, the coefficient merely changes sign;

(*b*) i is odd or even according as A contains odd or even powers of α;

(*c*) q is even in the expressions for $u\zeta^{-1}$, $s\zeta$ and odd in that for $z\iota$;

(*d*) A is of the order $\mathrm{e}^{|p|}\,e'^{|r|}\,\mathrm{k}^{|q|}$ at least, and it contains higher powers of e, e', k which differ from $|p|$, $|r|$, $|q|$ by even integers.

14. I now give the general notation adopted throughout. It is devised so as to represent every part of every coefficient.

The general term in $u\zeta^{-1}$ *or* $z\iota$ *will be expressed by*

$$\mathrm{a}(\epsilon^{p+p'}\epsilon'^{p'}\eta^{r+r'}\eta'^{r'}k^{q+q'}k'^{q'}\alpha^{s'})_i \mathrm{e}^{p+2p'}e'^{r+2r'}\mathrm{k}^{q+2q'}\alpha^{s'}\zeta^{2i\pm p\mathrm{c}\pm r\mathrm{m}\pm q\mathrm{g}} \quad \ldots \quad \ldots \quad (10)$$

where *

$$p,\ q,\ r,\ p',\ q',\ r',\ s'=0,\ 1,\ 2,\ \ldots,$$
$$2i=0,\ \pm1,\ \pm2,\ \ldots$$

The coefficient $(\quad)_i$ *above written corresponds to the upper signs in the index of* ζ.

With the lower sign of pc, *interchange the indices of* ϵ, ϵ';

With the lower sign of rm, *interchange the indices of* η, η';

With the lower sign of qg, *interchange the indices of* k, k'.

The sum of all such terms for all values of p, q, r, p', q', r', s', $2i$ *gives the complete expression of* $u\zeta^{-1}$ *or of* $z\iota$.

From the properties (a), (b), (c), (d) just given, it is evident *that when* q *is even,* (10) *gives the general term of* $u\zeta^{-1}$; *that when* q *is odd* (10) *gives the general term of* $z\iota$, *and that then*

$$(\epsilon^{p+p'}\epsilon'^{p'}\eta^{r+r'}\eta'^{r'}k^{q+q'}k'^{q'}\alpha^{s'})_i = -(\epsilon^{p'}\epsilon'^{p+p'}\eta^{r'}\eta'^{r+r'}k^{q'}k'^{q+q'}\alpha^{s'})_{-i};$$

that $s\zeta$ *is deduced from* $u\zeta^{-1}$ *by putting* $1/\zeta$ *for* ζ; *that* s', $2i$ *are odd or even together.*

When $2i$ is odd, we shall frequently denote this fact by putting $2i=2i_1$, so that i_1 will denote an odd multiple of $\pm\frac{1}{2}$. When an index of any symbol inside $()_i$ is zero, the symbol is simply omitted.

In the cases of the coefficients of the first order, namely, $(\epsilon)_i$, $(\epsilon')_i$, $(\eta)_i$, $(\eta')_i$, $(k)_i$, $(k')_i$, the brackets ( ) will be omitted for the sake of brevity, as they are unnecessary.

Particular Case.—In the case where p, q, ... s' are all zero no letter would occur inside $(\ )_i$. This being inconvenient we shall denote the corresponding coefficient by a_i. Thus, *the terms independent* of e, e' k, α *in* $u\zeta^{-1}$ *are denoted by*

$$\mathrm{a}\Sigma_i a_i \zeta^{2i}, \qquad i=0,\ \pm1,\ \pm2,\ \ldots$$

These are the terms of order zero (§ 15 below). There are no such terms in z.

* No confusion will be caused by this new use of the letters r, r', since they only occur, in this sense, in the indices of η, η', e', and have positive integral values.

15. *Characteristic and Order.*—The factor

$$\mathrm{e}^{p+2p'}e'^{r+2r'}\mathrm{k}^{q+2q'}\alpha^{s'}$$

of the general coefficient will be called the characteristic part of the coefficient or, briefly, the *characteristic.*

The *Order* is the sum of the indices of e, e', k, α. The order of the general term is therefore

$$p+2p'+r+2r'+q+2q'+s'.$$

The word "order," as used here, is thus independent of m—a necessary restriction, since the numerical value of m is substituted at the outset.

A few remarks and an example may make the notation laid down in § 14 clearer. It will be observed that

ϵ, ϵ' are always associated with e ;
η, η' are always associated with e' ;
k, k' are always associated with k ;
α is always associated with $\alpha=\mathrm{a}/a'$;

while, as is well known, e is associated with the index c ; e' with the index m ; k with the index g ; and an odd power of α with an odd value of $2i$.

If the numerical value of m had *not* been substituted at the outset, we could further have denoted the particular power of m involved by inserting m^j inside $(\ )_i$; the coefficient $(\ )_i$ would then have been a positive power series in m, with a numerical factor for each power of m, which factor is always the ratio of two integers.

The actual arrangement of $\epsilon^{p+p'}$, $\epsilon'^{p'}$, . . . inside $(\ )_i$ is immaterial, but we shall, in general, retain the arrangement of § 14.

Example.—The numerical part of the coefficient of $\zeta^{3+2\mathrm{m}-2\mathrm{g}}$ which has the characteristic $e'^4\mathrm{k}^2\alpha$ in $u\zeta^{-1}$ is denoted by

$$(\eta^3\eta' k'^2\alpha)_{\frac{3}{2}}.$$

For here, $p=0$, $p'=0$; $r=2$, $r'=1$; $q=2$, $q'=0$ with the indices of k, k' interchanged ; $2i=3$; $s'=1$. The series of terms in $u\zeta^{-1}$ which will be found along with this term, are those of the same characteristic which are

obtained by putting ζ^{-1} for ζ and whose indices differ from the given index of ζ by even whole numbers. See § 27 below. They are

$$\mathrm{a}e'^4\mathrm{k}^2\alpha\Sigma_{i_1}[(\eta^3\eta' k'^2\alpha)_{i_1}\zeta^{2i_1+2\mathrm{m}-2\mathrm{g}}+(\eta\eta'^3k^2\alpha)_{i_1}\zeta^{2i_1-2\mathrm{m}+2\mathrm{g}}],$$

where

$$2i_1=\pm 1,\ \pm 3,\ \pm 5,\ \ldots$$

16. *Notation for terms with a given Characteristic or of a given Order.*—It will frequently be convenient to specify such terms in a brief manner; this may be done by means of a suffix attached to u, s, z. Thus all terms in u with characteristic e^2 may be denoted by u_{e^2}, those with characteristic $e'\alpha$ by $u_{e'\alpha}$, and generally, those with characteristic λ by u_λ.

The terms of a given order are denoted by numerical suffixes. Thus, u_3 denotes all terms of the third order; u_0, those of zero order; and so on.

Section (v). *Method of Solution. Preparation of the Equations.*

17. The general feature of the method consists in the successive determination of the terms of orders 0, 1, 2, . . . with respect to powers and products of e, e', k, α in the coordinates u, z. As will be seen from the results of Section (ii), there are two methods of procedure—one by the use of the equations (3), (4), and the other by the use of the homogeneous equations (6), (7), (8). At the same time it may be stated that we need by no means confine ourselves to either of the two sets of equations, but use one or the other or both as may seem most convenient for the particular class of inequalities under consideration at any stage of the approximations.

18. *Terms of Order Zero.*—These terms, the coefficients of which are functions of m only, constitute a closed orbit with reference to the moving axes which is really the primary or "intermediate" orbit in the same sense as the elliptic orbit of the older theories; it may be called the *Variation Curve* since its principal periodic term is known as the "Variation." According to the notation of the last Section it is given by

$$u_0\zeta^{-1}=\mathrm{a}\Sigma a_i\zeta^{2i},\qquad s_0\zeta=\mathrm{a}\Sigma a_{-i}\zeta^{2i}.\qquad \ldots\quad \ldots\quad \ldots\qquad (11)$$

These values u_0, s_0 of u, s constitute a particular solution of the equation

$$(D+\mathrm{m})^2u+\frac{1}{2}\mathrm{m}^2u+\frac{3}{2}\mathrm{m}^2s-\frac{\kappa u}{\rho^3}=0\qquad \ldots\quad \ldots\quad \ldots\qquad (12)$$

or, in the homogeneous form, of the equations,

$$\left.\begin{array}{r} D^2(us)-Du\,.\,Ds-2\mathrm{m}(uDs-sDu)+\frac{9}{4}\mathrm{m}^2(u+s)^2=C' \\ D(uDs-sDu-2\mathrm{m}us)+\frac{3}{2}\mathrm{m}^2(u^2-s^2)=0 \end{array}\right\} \quad \ldots \quad (12')$$

since here, $\Omega_1=0$, $z=0$. The constant C'' is a function of a, m, while a is a function of $E+M$, n, m.

The values of u, s being substituted in the equations of either form (by preference, the latter), we obtain a number of equations of condition which suffice to determine $\mathrm{a}a_i$. In consequence of the presence of a, one of the a_i is arbitrary; we put $a_0=1$. The whole theory of these terms and the numerical results have been fully worked out by Dr. HILL in his paper "Researches in the Lunar Theory" (*Amer. Jour. Math.* vol. i.); the results will be given in Chap. II.

19. *Terms of the First Order.*—We put

$$u=u_0+u_1, \qquad z=z_1,$$

where

$$u_1=u_e+u_{e'}+u_a, \qquad z_1=z_k,$$

and neglect powers and products u_1, s_1, z_1, above the first. In the general equations of Section (ii) we put

$$\begin{array}{ll} \Omega_1=0,\ z=0, & \text{when } u_1=u_e\,; \\ \Omega_1=\omega_2,\ z=0, & \text{when } u_1=u_{e'}\,; \\ \Omega_1=\omega_3,\ z=0, & \text{when } u_1=u_a\,; \\ \Omega_1=0,\ u_1=0, & \text{when } z_1=z_k\,. \end{array}$$

Further, when $\Omega_1=\omega_2$ we neglect e'^2 and higher powers, and when $\Omega_1=\omega_3$ we put $e'=0$; in both cases u_0, s_0 may be put for u, s in Ω_1.

The right-hand members of the equations thus consist entirely of known terms. Putting $u=u_0+u_1$, $s=s_0+s_1$, $z=z_1$, and expanding the last terms of the left-hand members, it is easily seen that the equations (3), (4) for u_1, z_1 become

$$\left.\begin{array}{r} \zeta^{-1}(D+\mathrm{m})^2u_1+Mu_1\zeta^{-1}+Ns_1\zeta=\mathrm{a}\lambda A \\ D^2z_1-2Mz_1=0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where, putting $\rho_0{}^2=u_0s_0$,

$$\left.\begin{array}{l} M=\Sigma_i M_i\zeta^{2i}=\frac{1}{2}\mathrm{m}^2+\frac{1}{2}\frac{\kappa}{\rho_0{}^3} \\ N=\Sigma_i N_i\zeta^{2i}=\frac{3}{2}\mathrm{m}^2+\frac{3}{2}\frac{\kappa u_0{}^2\zeta^{-2}}{\rho_0{}^3} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and A consists of a ζ-series with known coefficients, and $\lambda=e'$ or α. When $u_1=u_e$, we have $A=0$. The sets of inequalities corresponding to u_e, $u_{e'}$, u_α, z_k can evidently be separately determined. In each case the appropriate expressions for u_1, s_1, z_1, are substituted, and equations of condition for the unknown coefficients are obtained by equating the coefficients of the various powers of ζ to zero; they are then solved by continued approximation.

In the equation for u_e, the questions of the arbitrary constant of eccentricity and of the motion of the perigee arise, and in that for z_k, the arbitrary constant of latitude and the motion of the node; these will be dealt with in their proper place (§§ 25, 26).

The homogeneous forms of the equations may be considered in a similar manner. I do not give the developments here as they may be easily inferred from the more general treatment in § 22 of this Section. Further information is given in chapter xi. of the "Treatise on the Lunar Theory" and in the papers on the Parallactic and Elliptic Inequalities (*Amer. Jour. Math.* vols. xiv., xv.).

The numerical results for the terms of the first order are contained in Chapter III below.

20. *Terms of the Second and Higher Orders. Development of the Equations* (3), (4).—The terms of the first order having been obtained, we proceed to show generally how the terms of any order and characteristic may be found when those of lower orders have been calculated. We shall deal with both sets of equations as either may be useful in certain cases. In this section the equations (3), (4) are considered.

They may be written,

$$\zeta^{-1}(D+\mathrm{m})^2u+\frac{1}{2}\mathrm{m}^2u\zeta^{-1}+\frac{3}{2}\mathrm{m}^2s\zeta\,.\,\zeta^{-2}-\frac{\kappa u\zeta^{-1}}{r^3}=-\zeta^{-1}\frac{\partial\Omega_1}{\partial s} \quad \ldots \quad (15)$$

$$(D^2-\mathrm{m}^2)z-\frac{\kappa z}{r^3}=-\frac{1}{2}\frac{\partial\Omega_1}{\partial z} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Suppose that it be required to determine all terms having the characteristic λ, say u_λ, in u, and those of characteristic* λ, say z_λ, in z.

* Of course λ can never be the same in the z-equation as it is in the u-equation. Consequently, as terms of different characteristics are never found in the same set of equations of condition, the equations (15), (16), are never considered *together* in finding any particular set of coefficients.

We put

$$u = u_0 + \Sigma u_\mu + u_\lambda, \qquad z = \Sigma z_\mu, \qquad \text{in (15)},$$
$$u = u_0 + \Sigma u_\mu, \qquad z = \Sigma z_\mu + z_\lambda, \qquad \text{in (16)}.$$

Here Σu_μ, Σz_μ contain all terms in u, z, *except* u_0, u_λ, z_λ, which will contribute to give terms with characteristic λ. These expressions are substituted in the equations, which are then expanded according to powers of $u-u_0$, z, all powers of u_λ, z_λ above the first and products of u_λ, z_λ with u_μ, z_μ being, of course, omitted. It will be remembered that $r^2 = us + z^2$.

Choosing out the terms which may produce terms with the characteristic λ we find

$$\begin{aligned}
&\zeta^{-1}(D+\mathrm{m})^2 u_\lambda + M u_\lambda \zeta^{-1} + N s_\lambda \zeta \\
&= \text{Part, char}^{\text{c}}.\ \lambda, \text{ in } \Big[-\zeta^{-1}(D^2 + 2\mathrm{m}D)\Sigma u_\mu - \frac{\partial \Omega_1}{\partial s}\zeta^{-1} \\
&\quad + \frac{\kappa u_0 \zeta^{-1}}{\rho_0^3} \Big\{ \frac{3}{8}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 + \frac{15}{8}\left(\frac{\Sigma s_\mu}{s_0}\right)^2 + \frac{3}{4}\frac{\Sigma u_\mu \,.\, \Sigma s_\mu}{u_0 s_0} - \frac{3}{2}\left(\frac{\Sigma z_\mu}{\rho_0}\right)^2 \\
&\quad - \frac{5}{16}\left(\frac{\Sigma u_\mu}{u_0}\right)^3 - \frac{35}{16}\left(\frac{\Sigma s_\mu}{s_0}\right)^3 - \frac{9}{16}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 \frac{\Sigma s_\mu}{s_0} - \frac{15}{16}\left(\frac{\Sigma s_\mu}{s_0}\right)^2 \frac{\Sigma u_\mu}{u_0} \\
&\qquad\qquad + \frac{9}{4}\left(\frac{\Sigma z_\mu}{\rho_0}\right)^2 \frac{\Sigma u_\mu}{u_0} + \frac{15}{4}\left(\frac{\Sigma z_\mu}{\rho_0}\right)^2 \frac{\Sigma s_\mu}{s_0} \\
&\quad + \ldots\ldots\ldots\ldots\ldots \Big\}\Big] \qquad \ldots \quad \ldots \quad \ldots \quad (17)
\end{aligned}$$

$$\begin{aligned}
&= \text{Part, char}^{\text{c}}.\ \lambda, \text{ in } \Big[-\zeta^{-1}(D^2 + 2\mathrm{m}D)\Sigma u_\mu - \frac{\partial \Omega_1}{\partial s}\zeta^{-1} \\
&\quad + \frac{1}{\mathrm{a}} \Big\{ \frac{3}{8}\overline{P}(\Sigma u_\mu \zeta^{-1})^2 + \frac{15}{8} Q(\Sigma s_\mu \zeta)^2 + \frac{3}{4} P \Sigma u_\mu \,.\, \Sigma s_\mu - \frac{3}{2} P(\Sigma z_\mu)^2 \Big\} \\
&\quad - \frac{1}{\mathrm{a}^2} \Big\{ \frac{5}{16}\overline{R}(\Sigma u_\mu \zeta^{-1})^3 + \frac{35}{16} T(\Sigma s_\mu \zeta)^3 + \frac{9}{16} S(\Sigma u_\mu \zeta^{-1})^2 \Sigma s_\mu \zeta + \frac{15}{16} R(\Sigma s_\mu \zeta)^2 \Sigma u_\mu \zeta^{-1} \\
&\qquad\qquad - \frac{9}{4} S(\Sigma z_\mu)^2 \Sigma u_\mu \zeta^{-1} - \frac{15}{4} R(\Sigma z_\mu)^2 \Sigma s_\mu \zeta \Big\} \\
&\quad + \ldots\ldots\ldots\ldots\ldots \Big] \ldots \quad \ldots \quad (17')
\end{aligned}$$

$$\begin{aligned}
D^2 z_\lambda - 2 M z_\lambda = \text{Part, char}^{\text{c}}.\ \lambda, \text{ in } \Big[&-D^2 \Sigma z_\mu - \frac{1}{2}\frac{\partial \Omega_1}{\partial z} \\
&+ \frac{\kappa}{\rho_0^2}\Big[-\frac{3}{2}\frac{\Sigma z_\mu}{\rho_0}\left(\frac{\Sigma u_\mu}{u_0} + \frac{\Sigma s_\mu}{s_0}\right) \\
+ \frac{\Sigma z_\mu}{\rho_0} \Big\{ \frac{15}{8}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 + \frac{15}{8}\left(\frac{\Sigma s_\mu}{s_0}\right)^2 &+ \frac{9}{4}\frac{\Sigma u_\mu}{u_0}\frac{\Sigma s_\mu}{s_0} \Big\} - \frac{3}{2}\left(\frac{\Sigma z_\mu}{\rho_0}\right)^3 \\
- \ldots\ldots\ldots\ldots\ldots &\Big]\Big] \ldots \quad \ldots \quad (18)
\end{aligned}$$

$$=\text{Part, char}^{\text{c}}.\ \lambda,\ \text{in}\left[-D^2\Sigma z_\mu-\frac{1}{2}\frac{\partial\Omega_1}{\partial z}\right.$$

$$-\frac{3}{2}\frac{\Sigma z_\mu}{\mathrm{a}}(P\Sigma u_\mu\zeta^{-1}+\bar{P}\Sigma s_\mu\zeta)$$

$$+\frac{\Sigma z_\mu}{\mathrm{a}^2}\left\{\frac{15}{8}\bar{R}(\Sigma u_\mu\zeta^{-1})^2+\frac{15}{8}R(\Sigma s_\mu\zeta)^2+\frac{9}{4}S\Sigma u_\mu\,.\,\Sigma s_\mu\right\}-\frac{3}{2}S\frac{(\Sigma z_\mu)^3}{\mathrm{a}^2}$$

$$\left.-\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \right]\quad\ldots\quad\ldots\quad(18')$$

where

$$\left.\begin{array}{lll} P=\Sigma_i P_i\zeta^{2i}=\mathrm{a}\dfrac{\kappa u_0\zeta^{-1}}{\rho_0^{\,5}}, & Q=\Sigma_i Q_i\zeta^{2i}=\mathrm{a}\dfrac{\kappa u_0^{\,3}\zeta^{-3}}{\rho_0^{\,7}}, & \\ R=\Sigma_i R_i\zeta^{2i}=\mathrm{a}^2\dfrac{\kappa u_0^{\,2}\zeta^{-2}}{\rho_0^{\,7}}, & T=\Sigma_i T_i\zeta^{2i}=\mathrm{a}^2\dfrac{\kappa u_0^{\,4}\zeta^{-4}}{\rho_0^{\,9}}, & \\ S=\Sigma_i S_i\zeta^{2i}=\mathrm{a}^2\dfrac{\kappa}{\rho_0^{\,5}}, & \bar{P}=\Sigma_i P_{-i}\zeta^{2i}, & \bar{R}=\Sigma_i R_{-i}\zeta^{2i},\ \text{etc.} \end{array}\right\}\quad\ldots\quad\ldots\quad(19)$$

the values of $\bar{P}$, $\bar{R}$ being obtained from those of P, R by interchanging $u_0\zeta^{-1}$, $s_0\zeta$, that is, by putting $1/\zeta$ for ζ.*

21. Some remarks on these equations are necessary. In the first place, the left-hand members of (3), (4) being linear with respect to u, s, and to z respectively (exception being made of the terms containing κ; these will be considered immediately), the parts $u_0+\Sigma u_\mu$ of u and Σz_μ of z cannot contribute anything to terms with characteristic λ as far as the *coefficients* of these periodic terms are concerned. But the operators D^2, D cause the coefficients in u_μ, z_μ to be respectively multiplied by factors of the forms

$$(2i+p\mathrm{c}+r\mathrm{m}+q\mathrm{g})^2,\qquad(2i+p\mathrm{c}+r\mathrm{m}+q\mathrm{g}),$$

and c, g contain powers and products of e^2, e'^2, k^2, α^2. Hence it will be necessary in some of the terms, whose orders are higher than the second, to include D^2u_μ, Du_μ, D^2z_μ in the equations.

In general, all the unknowns are contained in the left-hand members of (17) or (17′) and (18) or (18′), while the terms of the right-hand members are entirely known. Exception to the last statement only holds when we are determining an unknown part of c or g. It must be remembered that, in

* In the *Investigations*, p. 327, where these expressions are used, one or two errors must be noted. The factor a (there called a_0) is omitted from the values of P, $\bar{P}$, Q and the factor a^2 from those of R, $\bar{R}$, T, S. The notations for R, $\bar{R}$ should be interchanged, that is R_{-i} put for R_i, to bring them into uniformity with those for P, $\bar{P}$.

reality, c and g are supposed to represent the complete values at the outset, but that, in forming the equations of condition, all terms of higher orders in the values of c, g, than those actually under consideration are neglected. Hence an unknown part of c or g will, in certain cases, arise from the terms containing the operator D in the right-hand members. These cases are more fully considered in the next Section.

As to the terms involving $\mathfrak{S}_1$, since $\mathfrak{S}_1$ is of the first order at least, apart from the order of the terms in u, we can evidently substitute $u_0+\Sigma u_\mu$ for u and Σz_μ for z therein.

The rest of the terms arise from the expansions of $\frac{\kappa u\zeta^{-1}}{r^3}$, $\frac{\kappa z}{r^3}$ in powers of Σu_μ, Σz_μ. Those containing the first powers of these quantities are omitted, since they evidently cannot produce terms with the characteristic λ. It will be noticed that the terms are written in two ways, (17), (18) and (17′), (18′).

If we take the first forms, namely (17), (18), for calculation, the process is to calculate each $\Sigma u_\mu/u_0$, $\Sigma z_\mu/\rho_0$ (which consists in a single easy multiplication of each $u_\mu\zeta^{-1}$ by $s_0\zeta/\rho_0^2$ and of each z_μ by $1/\rho_0$), to form the various products inside the parentheses, and finally to multiply the resulting series by $\kappa u_0\zeta^{-1}/\rho_0^3$ and κ/ρ_0^2 in the respective equations. For this purpose the values of

$$\mathrm{a}\frac{s_0\zeta}{\rho_0^2}, \qquad \frac{\mathrm{a}}{\rho_0}, \qquad \frac{\kappa u_0\zeta^{-1}}{\mathrm{a}\rho_0^3}, \qquad \frac{\kappa}{\mathrm{a}\rho_0^2},$$

expressed as even-power series in ζ are given at the end of Chapter II.

If we take the second forms, namely, (17′), (18′), the various multiplications which have to be made are evident, and the values of P, Q, . . . necessary for this, expressed as power series in ζ, are given in the same place. There can be little doubt that the first forms give shorter calculations, and they have a further advantage in the fact that the only trouble necessary for transferring to polar coordinates consists in the calculations of powers of $\Sigma u_\mu/u_0$, $\Sigma z_\mu/\rho_0$ (see Section ix), and this labour will therefore have already been finished. Indeed, the second forms would not have been given here at all, were it not for the fact that I failed to see the great advantage of the first forms for the higher inequalities, and consequently used the second forms in

the calculation of all the second order inequalities and of a few of the third order.*

A great advantage of these equations is that the chief labour—the multiplication of series—can be easily arranged for an ordinary computer, and much of the merely mechanical labour may thereby be distributed (see Sect. viii).

The numerical results for terms of the second order are given in Chapter IV. below; those for terms of the third and higher orders will be given in chapters to be published hereafter.

22. *First and Higher Order Terms. Homogeneous Equations.*—The separation of the homogeneous equations into known and unknown parts is effected in a similar manner. The substitutions for u, z are the same as in § 20. If the homogeneous equations be used for actually finding the coefficients, the forms obtained in equations (20), (21), (22) are of no assistance, except as a guide; this will be evident by a glance at §§ 32–36. They have been, however, almost exclusively used for verifying the results obtained from the equations of § 20, and they are given mainly for that purpose here.

They are

$$\begin{aligned} D^2(u_0 s_\lambda + s_0 u_\lambda) - Du_0 Ds_\lambda - Ds_0 Du_\lambda - 2\mathrm{m}(u_0 Ds_\lambda - s_0 Du_\lambda + u_\lambda Ds_0 - s_\lambda Du_0) \\ + \frac{9}{2}\mathrm{m}^2(u_0 + s_0)(u_\lambda + s_\lambda) \end{aligned}$$

$$\begin{aligned} =\text{Part, char}^{\text{c}}.\ \lambda,\ \text{in} \Big[C' - \sum_{q=2}^{\infty}(q+1)\omega_q + D^{-1}(D'\Omega_1) \\ - \Big\{ D^2(\Sigma u_\mu \,.\, \Sigma s_\mu) - D\Sigma u_\mu \,.\, D\Sigma s_\mu - 2\mathrm{m}(\Sigma u_\mu D\Sigma s_\mu - \Sigma s_\mu D\Sigma u_\mu) + \frac{9}{4}\mathrm{m}^2(\Sigma u_\mu + \Sigma s_\mu)^2 \\ - (D\Sigma z_\mu)^2 - \frac{3}{2}\mathrm{m}^2(\Sigma z_\mu)^2 \Big\} \Big] \quad \ldots \quad \ldots \quad \ldots \quad (20) \end{aligned}$$

$$D(u_0 Ds_\lambda - s_0 Du_\lambda + u_\lambda Ds_0 - s_\lambda Du_0 - 2\mathrm{m}u_0 s_\lambda - 2\mathrm{m}s_0 u_\lambda) + 3\mathrm{m}^2(u_0 u_\lambda - s_0 s_\lambda)$$

$$\begin{aligned} =\text{Part, char}^{\text{c}}.\ \lambda,\ \text{in} \Big[s\frac{\partial\Omega_1}{\partial s} - u\frac{\partial\Omega_1}{\partial u} \\ - \Big\{ D(\Sigma u_\mu \,.\, D\Sigma s_\mu - \Sigma s_\mu \,.\, D\Sigma u_\mu - 2\mathrm{m}\Sigma u_\mu \,.\, \Sigma s_\mu) + \frac{3}{2}\mathrm{m}^2(\Sigma u_\mu)^2 - \frac{3}{2}\mathrm{m}^2(\Sigma s_\mu)^2 \Big\} \Big] \quad \ldots \quad \ldots \quad (21) \end{aligned}$$

* From the remarks just made, it might have seemed an improvement to put $u = u_0 v$, $s = s_0 \bar{v}$, $z = \rho_0 w$ and to find v, $\bar{v}$, w only. This, however, only throws some of the labour of forming u_μ / u_0, etc., on to the solution of the equations of condition; the latter process is far less mechanical than the former and much more liable to error, and there will be no saving of labour.

$$D(u_0 Dz_\lambda - z_\lambda Du_0) - 2\text{m}z_\lambda Du_0 - \text{m}^2 u_0 z_\lambda - \frac{3}{2}\text{m}^2 z_\lambda(u_0 + s_0)$$
$$= \text{Part, char}^{\text{c}}.\ \lambda,\ \text{in} \left[z\frac{\partial \Omega_1}{\partial s} - \frac{1}{2}u\frac{\partial \Omega_1}{\partial z} \right.$$
$$- \{D(\Sigma u_\mu \,.\, D\Sigma z_\mu - \Sigma z_\mu \,.\, D\Sigma u_\mu) - 2\text{m}\Sigma z_\mu \,.\, D\Sigma u_\mu - \text{m}^2\Sigma u_\mu .\, \Sigma z_\mu$$
$$\left. - \frac{3}{2}\text{m}^2\Sigma z_\mu(\Sigma u_\mu + \Sigma s_\mu)\} \right] \qquad \ldots \qquad \ldots \qquad (22)$$

As before, the left-hand members contain the unknown terms. The right-hand members consist entirely of known terms, except when new parts of the motions of perigee or node are under consideration. In certain cases it will be necessary to suppose u_0, s_0 to be included in Σu_μ, Σs_μ ; see the remarks at the beginning of § 21. In the terms involving Ω_1 ; we may substitute $u_0 + \Sigma u_\mu$, Σz_μ for u, z, respectively ; the method by which the calculations are to be actually performed will be given in Section (vii).

In the case of the first order terms, Σu_μ, Σz_μ are both zero, and the limitations of Ω_1 are the same as those given in § 19.

Section (vi). *Definitions of the Arbitrary Constants.*

23. Of the six arbitrary constants of the solution, three have already been defined, namely, the three angular constants in the arguments D, l, F. A fourth, n, has also been defined as the observed mean motion. It remains to give an exact definition to a, the linear constant (which replaces the mass $E+M$), and to e, k, the constants of eccentricity and inclination.

24. *Definition of* a.—In elliptic motion, a is defined by means of the relation $n^2\text{a}^3 = E+M$. For many purposes this is much the most convenient definition even when we proceed to determine the solar inequalities in the lunar theory. But in the theory developed here, the calculations may be materially shortened by a different definition of a.

The value of u_0 is given by

$$u_0\zeta^{-1} = \text{a}\Sigma_i a_i \zeta^{2i}, \qquad i = 0,\ \pm 1,\ \pm 2,\ \ldots$$

From the form of this it is evident that either a or a_0 may be chosen to be anything we wish. The most convenient definition is obtained by putting

$$a_0 = 1,$$

so that

$$a=\left(\frac{E+M}{n^2}\right)^{\frac{1}{3}}f(m) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where f (m) is a function of m which, in the case of the Moon, is very nearly unity.

The definition must now be extended so as to cover the case when inequalities of any order are being considered. The general form of all the inequalities which have arguments of the form ζ^{2i} ($2i$ even), are given by (§ 14)

$$u\zeta^{-1}=a\Sigma_i[a_i+\Sigma(\epsilon^{p'}\epsilon'^{p'}\eta^{r'}\eta'^{r'}k^{q'}k'^{q'}\alpha^{2s'})_i e^{2p'}e'^{2r'}k^{2q'}\alpha^{2s'}]\zeta^{2i}$$

where

$$i=0,\ \pm1,\ \pm2,\ \ldots;\quad p',\ q',\ r',\ s'=0,\ 1,\ 2,\ \ldots\quad (\text{except } p'=q'=r'=s'=0);$$

and $\Sigma\ (\ )_i$ denotes the sum of all such terms for the values of p', q', r', s', given. The coefficient of ζ^0 in $u\zeta^{-1}$, by means of which a is to be defined, is therefore, since $a_o=1$,

$$a[1+\Sigma(\epsilon^{p'}\epsilon'^{p'}\eta^{r'}\eta'^{r'}k^{q'}k'^{q'}\alpha^{2s'})_0 e^{2p'}e'^{2r'}k^{2q'}\alpha^{2s'}],$$
$$=a(1+\nu),\qquad \text{suppose},$$

so that ν is a small quantity of the second order at least.

There are two practical methods of defining a, each of which has its use according to the equations we employ.

One of these is to so define a that every term in ν is zero, and therefore that the coefficient of ζ^0 is represented by a at every stage of the approximations. This definition requires the determination of some additional terms to a whenever we are finding inequalities of the form ζ^{2i}. If we are using the homogeneous equations, this is undoubtedly the best definition, for then the determination of the additional terms in a can be left till the end of the work, and as a only appears in the parallax and not in the longitude and latitude, a very great degree of accuracy in its value is not required. In using the equations (3), (4), however, it would cause inconvenience as we should then have to find a further approximation to a at each step. As the latter equations are those mainly used here, we shall adopt another definition better adapted to the calculations.

This second definition (which we shall use below) is to give to a the meaning which it receives from the intermediate orbit only and to retain it throughout. Thus a is defined by the equation (23) making $a\{(E+M)/n^2\}^{-\frac{1}{3}}$

a numerical constant which never alters. *The coefficients* ν *are now no longer zero*, but are definite functions of e^2, e'^2, k^2, a^2, being determined along with the other coefficients of ζ^{2i} in the ordinary way. This definition is necessary because of the want of homogeneity of the equations (3), (4).* In finding the parallax from the value of u, all that will be necessary will be to find a/r and then to multiply all the terms by a numerical quantity (which approaches unity very closely) in order to obtain $[(E+M)/n^2]^{\frac{1}{3}}/r$—the quantity usually obtained by lunar theorists.

Hence, *the linear constant* a *is defined to be the coefficient of* ζ^0 *in* $u_0\zeta^{-1}$, *where* $u_0\zeta^{-1}$ *represents the intermediate orbit or variation curve only.* Its value is given in Chap. II., and it retains this value throughout the whole of the approximations.

25. *Definition of* e. The first of equations (13) for the determination of the inequalities depending on the first power of e is, since $A=0$,

$$\zeta^{-1}(D+\mathrm{m})^2u_e+Mu_e\zeta^{-1}+Ns_e\zeta=0,$$

the solution of which is obtained by assuming

$$u_e\zeta^{-1}=\mathrm{ae}\Sigma_i(\epsilon_i\zeta^{2i+c}+\epsilon'_i\zeta^{2i-c}),\qquad i=0,\ \pm1,\ \pm2,\ \ldots$$

When the substitution is made, and the coefficients of the various powers of $\zeta^{2i\pm c}$ equated to zero, we obtain a series of equations of condition for the determination of the unknowns ϵ_i, ϵ'_i, e, which are homogeneous and of the first degree with respect to ϵ_i, ϵ'_i. The determination of e is made by considering the necessary relation which must exist between these equations; it is actually found by a different method (see § 28 (*b*)), and we suppose it known. One of the $\mathrm{e}\epsilon_i$, $\mathrm{e}\epsilon'_i$, is arbitrary. The values of ϵ_i, ϵ'_i may all be made definite by taking e as the arbitrary constant and putting

$$\epsilon_0-\epsilon'_0=1.$$

* This is the definition intended in the remarks on p. 343 of the *Investigations*. A want of clearness in the statement of the definition in that paper has caused a misapprehension of its meaning. Mr. P. H. Cowell in his paper "On the Inclinational Terms" (*Amer. Jour. Math.* vol. xviii.) uses the homogeneous equations for the determination of the coefficients $(kk')_i$, and naturally finds it more convenient to put $(kk')_0=0$. He, however, does not find the addition to the value of a (there called a_0); this requires a reference to one of the equations containing κ. The definition is, I hope, made quite clear in § 24 above.

The coefficients of ζ^c, ζ^{-c} in $u\zeta^{-1}$ are $\mathrm{ae}\epsilon_0$, $\mathrm{ae}\epsilon_0'$, respectively. Since, by equations (9),

$$u\zeta^{-1} = \rho \cos(V - nt - \epsilon) + \iota\rho \sin(V - nt - \epsilon),$$

the coefficient of a sin l in ρ sin ($V-nt-\epsilon$) will be *

$$\mathrm{e}(\epsilon_0 - \epsilon_0') = \mathrm{e},$$

by the use of the assumed relation. The value of e thus defined is very nearly twice the constant of eccentricity used by DELAUNAY.

The general form of all the terms in $u\zeta^{-1}$, which involve ζ^c, ζ^{-c}, is (§ 14),

$$\begin{aligned}\mathrm{ae}\Big[\Sigma_{p',q',r',s'}\{(\epsilon^{1+p'}\epsilon'^{p'}\eta^{r'}\eta'^{r'}k^{q'}k'^{q'}\alpha^{2s'})_0\zeta^c \\ + (\epsilon^{p'}\epsilon'^{1+p'}\eta^{r'}\eta'^{r'}k^{q'}k'^{q'}\alpha^{2s'})_0\zeta^{-c}\}\, \mathrm{e}^{2p'}e'^{2r'}\mathrm{k}^{2q'}\alpha^{2s'}\Big].\end{aligned}$$

The definition of e must be extended so as to cover all these terms. It has been found most convenient to define it to be such that the coefficients of ζ^c and ζ^{-c} in the above expression are equal, except when $p'=q'=r'=s'=0$, when we have already defined it by making their difference unity. Denoting, for a moment, each of these equal coefficients by β, so that the terms containing ζ^c, ζ^{-c} in $u\zeta^{-1}$ are given by

$$\mathrm{ae}(\epsilon_0\zeta^c + \epsilon_0'\zeta^{-c}) + \mathrm{ae}\beta(\zeta^c + \zeta^{-c}),$$

it is evident that the coefficient of a sin l in ρ sin ($V-nt-\epsilon$) will be $\mathrm{e}(\epsilon_0 - \epsilon_0') = \mathrm{e}$.

Hence *the constant of eccentricity* e *is defined to be the coefficient of* a sin l *in the final expression of* ρ sin ($V-nt-\epsilon$) *as a sum of periodic terms, where* $V-nt-\epsilon$ *is the difference of the true and mean longitudes and* ρ *is the projection of the Moon's radius vector on the plane of reference.*

26. *Definition of* k. The second of equations (13) for the determination of z_{k} is

$$D^2 z_{\mathrm{k}} - 2Mz_{\mathrm{k}} = 0,$$

in which we substitute

$$z_{\mathrm{k}}\iota = \mathrm{ak}\Sigma_i(k_i\zeta^{2i+g} + k'_i\zeta^{2i-g})$$

where $k'_i = -k_{-i}$.

* This definition, for the terms with characteristic e, is the same as that which I adopted in "The Elliptic Inequalities" (*Amer. Jour. Math.* vol. xv. p. 261); e is there called Y_0.

The constant k is now defined to be such that

$$k_0 = -k'_0 = 1,$$

so that 2k is the coefficient of a sin F in the expression of z as a sum of periodic terms. The constant k differs little from DELAUNAY's constant γ.

The general form of all inequalities containing only $\zeta^{\pm g}$ in $z\iota$ is

$$\mathrm{ak}\Sigma_{p', q', r', s'}(\epsilon^{p'}\epsilon'^{p'}\eta^{r'}\eta'^{r'}k^{1+q'}k'^{q'}\alpha^{2s'})_0 \mathrm{e}^{2p'}e'^{2r'}\mathrm{k}^{2q'}\alpha^{2s'}(\zeta^{g}-\zeta^{-g}),$$

and the definition of k must be extended so as to cover all these terms.

We now define it to be such that

$$(\epsilon^{p'}\epsilon'^{p'}\eta^{r'}\eta'^{r'}k^{1+q'}k'^{q'}\alpha^{2s'})_0 = 0,$$

for all values of p', q', r', s', except for $p'=q'=r'=s'=0$, when we have already put $()_0$ equal to unity. It is to be remembered that if we interchange the accents and the sign of i, the coefficient merely changes sign; hence the coefficient corresponding to that just written down is also zero.

Hence, *the constant of latitude* k *is defined to be the coefficient of* 2a sin F *in the expression of z as a sum of periodic terms.*

Section (vii).—*Solution of the Equations of Condition. Motions of the Perigee and Node.*

It will here be necessary to divide up the subject according as we are treating the equation (17) for u, the equation (18) for z, the homogeneous equations (20), (21) for u, or the homogeneous equation (22) for z.

27. *The Equation* (17) *for u.* The general type of equation for the terms of characteristic λ in u is

$$\zeta^{-1}(D+\mathrm{m})^2 u_\lambda + M u_\lambda \zeta^{-1} + N s_\lambda \zeta = \mathrm{a}\lambda A \quad \dots \quad \dots \quad \dots \qquad (24)$$

where it is to be remembered that M, N are known even-power series in ζ with numerical coefficients (equations (14)) and A contains the known terms with characteristic λ arising from the right-hand member. In only one case do the latter terms contain an unknown quantity; this case, which involves the determination of a part of the motion of the perigee, will be treated in § 28 (*b*).

Of the terms with characteristic λ, suppose that we require to know those in $u\zeta^{-1}$ which involve $\zeta^{2i\pm\tau}$, where τ is one of the values of

$$\pm pc \pm rm \pm 2qg,$$

the right-hand member containing such terms. Let, therefore, these terms in A be denoted by

$$A = \Sigma_i (A_i \zeta^{2i+\tau} + A'_i \zeta^{2i-\tau}).$$

We substitute

$$u_\lambda \zeta^{-1} = a\lambda \Sigma_i (\lambda_i \zeta^{2i+\tau} + \lambda'_i \zeta^{2i-\tau}). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

where λ_i, λ'_i are the unknown coefficients to be found, and equate to zero the coefficients of $\zeta^{2i+\tau}$, $\zeta^{-2i-\tau}$. The result is

$$\left.\begin{aligned} (2i+\tau+1+m)^2\lambda_i + \Sigma_j M_j \lambda_{i-j} + \Sigma_j N_j \lambda'_{j-i} &= A_i \\ (2i+\tau-1-m)^2\lambda'_{-i} + \Sigma_j M_j \lambda'_{-i-j} + \Sigma_j N_j \lambda_{j+i} &= A'_{-i} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (26)$$

where $j=0, \pm 1, \pm 2, \ldots$ and $2i$ either $=0, \pm 2, \pm 4, \ldots$ or $=\pm 1, \pm 3, \pm 5, \ldots$ Since τ multiplies λ_i and λ'_{-i} it is evident that the complete values of c, g on the left may be replaced by their values c_0, g_0 which are functions of m only.

The M_i, N_i have quickly decreasing values for increasing values of i (see Chapter II.) and, in general, the same remark applies to A_i, A'_{-i}, λ_i, λ'_{-i}. The equations may thus be solved by continued approximation. The unknown terms of lowest order in the equations (26) are

$$[(2i+\tau+1+m)^2 + M_0]\lambda_i + N_0 \lambda'_{-i},$$
$$N_0 \lambda_i + [(2i+\tau-1-m)^2 + M_0]\lambda'_{-i},$$

respectively. The equations (26) are therefore those of principal importance in finding λ_i, λ'_{-i}.

When we solve them as two simultaneous equations to find λ_i, λ'_{-i}, the common divisor is

$$[(2i+\tau+1+m)^2 + M_0][(2i+\tau-1-m)^2 + M_0] - N_0^2;$$

and, by the results contained in Part III. of the *Investigations*, it will be seen that this is very nearly equal to

$$(2i+\tau)^2[(2i+\tau)^2 - c_0^2] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

If we had eliminated all the other unknowns from the equations (26), this expression would have been a factor of the divisor, the other factor being very nearly unity. In considering the solution it is then only necessary to treat the cases where the expression (27) becomes small.

28. There are *four* special cases to consider—namely, the cases when either factor of (27) is zero or small. We recall (§ 14) that when inequalities involving odd powers of α are under consideration, $2i$ is an odd positive or negative integer. It is unnecessary to prove many of the statements made below; their truth, if not evident, can easily be demonstrated.

(*a*) *The case* $2i+\tau=0$. Here we must have $\tau=0$, $i=0$, owing to the incommensurability of c, g, m, 1. It is a question of determining coefficients of ζ^{2i}; there are no coefficients λ'_i and the two equations coalesce into one which is of principal importance in determining λ_0.

(*b*) *The case* $2i+\tau=\pm c_0$. *Motion of the Perigee.* Here $i=0$, $\tau=c_0$; $\tau=-c_0$ is the same case as $\tau=c_0$ as we consider ζ^c, ζ^{-c} together. In this case A contains an unknown quantity—namely, the part of the motion of the perigee which has the characteristic λ/e, say, $c_{\lambda/e}$, and it will be found possible to put

$$A_i=B_i+c_{\lambda/e}b_i, \qquad A'_i=B'_i+c_{\lambda/e}b'_i$$

where B_i, b_i, B'_i, b'_i are entirely known. It will be found also that b_i, b'_i are always the same whatever λ may be.

Substitute these values for A_i, A'_{-i} in (26). Multiply the first equation by ϵ_i, the second by ϵ'_{-i}, and take the sum for all values of i. Since $\tau=c_0$, we find

$$\begin{aligned}\Sigma_i[(2i+c_0+1+m)^2\lambda_i\epsilon_i+(2i+c_0-1-m)^2\lambda'_{-i}\epsilon'_{-i}]&\\ +\Sigma_i\Sigma_j M_j(\lambda_{i-j}\epsilon_i+\lambda'_{-i-j}\epsilon'_{-i})+\Sigma_i\Sigma_j N_j(\lambda'_{j-i}\epsilon_i+\lambda_{j+i}\epsilon'_{-i})&\\ =\Sigma_i[B_i\epsilon_i+B'_{-i}\epsilon'_{-i}+c_{\lambda/e}(b_i\epsilon_i+b'_{-i}\epsilon'_{-i})] \quad \ldots \quad \ldots \quad \ldots \quad \ldots &\quad (28)\end{aligned}$$

But for the terms with characteristic e, we have

$$\begin{gathered}(2i+c_0+1+m)^2\epsilon_i+\Sigma_j M_j\epsilon_{i-j}+\Sigma_j N_j\epsilon'_{j-i}=0,\\ (2i+c_0-1-m)\epsilon'_{-i}+\Sigma_j M_j\epsilon'_{-i-j}+\Sigma_j N_j\epsilon_{j+i}=0.\end{gathered}$$

Substituting these in the previous equation, it is easily seen* that the left-hand member vanishes and therefore the right-hand member of (28) equated to zero determines $c_{\lambda/e}$. (See *Investigations*, p. 336).

* For we have $\Sigma_i\Sigma_j M_j\lambda_{i-j}\epsilon_i=\Sigma_i\Sigma_j M_{-j}\lambda_i\epsilon_{i-j}$ and $M_{-j}=M_j$, etc.

When the value of this quantity has been found, the equations (26) may be solved by continued approximation, all the λ_i, λ'_i being expressed in terms of λ_0, λ'_0. One of the two equations for λ_0, λ'_0 can then be deduced from the other. An arbitrary relation connects the λ_i, λ'_i. We have already (§ 25) settled this relation to be such that $\lambda_0=\lambda'_0$. The determination of a new part of the motion of the perigee thus goes with a more accurate definition of the constant of eccentricity.

The manner in which c_0 may be best obtained is fully discussed by Dr. Hill in his paper "On the Part of the Motion of the Lunar Perigee, &c." (*Acta Math.* vol. viii.). Its value is there found to fifteen places of decimals. The parts of c which have the characteristics e^2, e'^2, k^2, α^2 properly belong to Chapter V. ("Inequalities of the Third Order") of this memoir. The calculations, however, have been advanced in this particular direction so that c_{e^2}, $c_{e'^2}$, c_{k^2} might be obtained; they will be found in the appendix to Chapter IV.

(*c*) *The case* $2i+\tau$ *small compared with unity. The inequalities which are of long period compared with the lunar month.* A troublesome defect of the method arises here. The divisor (27) contains the *square* of $2i+\tau$, while the corresponding coefficients λ_i, λ'_{-i} are in general of the same order of magnitude as $A_i/(2i+\tau)$. The reason of this is easily seen on solving the equations: one of them, in fact, generally differs from being deducible from the other by a quantity which is of the order of magnitude $2i+\tau$. This is illustrated in a striking manner by the long-period inequality whose argument is $2F-2l$ (which is one of the most troublesome in any method). Here $2i+\tau=2g_0-2c_0$ $=+\cdot 0272$, nearly, and therefore $(2i+\tau)^2=+\cdot 00074$, while the corresponding numerical coefficients are of the order of magnitude unity.

The difficulty in all these cases is best avoided by computing the homogeneous equation (21) for the particular value of $2i+\tau$, and combining it with one of the equations (26). In (21) the terms divided by $(2i+\tau)^2$ have the factor m^2, and therefore with the same degree of accuracy in the known parts of the equation we are able to obtain λ_i, λ'_{-i} more accurately than if we simply used the two equations (26).

(*d*) *The case* $2i+\tau\pm c_0$ *small compared with unity. The numerous short-period inequalities the mean motions of whose arguments approximate to that of the principal elliptic term*; *e.g.* the Evection and the Parallactic Inequality. The method is not in defect here, since the divisor is of the same numerical

magnitude as that arising in any other method. It will be noticed that the divisors are smaller according as the periods approach more nearly to that of the principal elliptic term and not to the lunar mean sidereal or synodic periods.

29. When the pair of equations for λ_i, λ'_{-i} possess a small divisor, the approximations proceed slowly. In some of these cases it is advisable to save labour by finding $\lambda_{i\pm 1}$, $\lambda'_{i\pm 1}$, in terms of λ_i, λ'_{-i} and the known quantities, from the equations with suffixes $\pm i \pm 1$, and to substitute the results in the equations with suffix i before solving the latter. In all cases where the difficulty occurred, it has been avoided by this device.

30. *The Equation* (18) *for* z. The course of the argument is very similar to that in §§ 27, 28, and therefore the results will be given briefly.

The general type of equation for the terms with characteristic λ in z is, by the equation (18),

$$D^2 z_\lambda - 2Mz_\lambda = \mathrm{a}\lambda A\iota \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where A represents the known terms. For the terms which involve $\zeta^{2i\pm\tau}$, we find

$$A = \Sigma_i A_i(\zeta^{2i+\tau} - \zeta^{-2i-\tau}) \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

and substitute

$$z_\lambda \iota = \mathrm{a}\lambda \Sigma_i \lambda_i(\zeta^{2i+\tau} - \zeta^{-2i-\tau}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

since in z we always have $\lambda'_{-i} = -\lambda_i$. The equations of condition are

$$(2i+\tau)^2\lambda_i - 2\Sigma_j M_j \lambda_{i-j} = A_i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where $j = 0, \pm 1, \pm 2, \ldots$ and $2i$ either $= 0, \pm 2, \pm 4, \ldots$ or $= \pm 1, \pm 3, \pm 5 \ldots$.

The equations (32) are solved by continued approximation—that written down being of principal importance in finding λ_i. The coefficient of λ_i is

$$(2i+\tau)^2 - 2M_0$$

which, if we had eliminated all the other unknowns, would have been

$$(2i+\tau)^2 - \mathrm{g}_0{}^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

multiplied by a numerical factor which is nearly unity.

31. There are only two cases to consider, namely, those in which $2i+\tau \pm \mathrm{g}_0$ is zero or small compared with unity. Hence no long-period inequalities can

give rise directly to large coefficients in z; thus the cases (a), (c) of § 28 do not arise; the cases corresponding to (b), (d) are those numbered (b'), (d') below.

(b') *The case* $2i+\tau=\pm g_0$. *Motion of the Node.* Here $i=0$, $\tau=\pm g_0$, and A contains an unknown part of the motion of the node of the form $g_{\lambda/k}$. We find

$$A_i=B_i+g_{\lambda/k}b_i,$$

where B_i, b_i are entirely known. Substituting for A_i in (32), multiplying the equation by k_i and summing for all values of i, we obtain

$$\Sigma_i(2i+g_0)^2\lambda_ik_i-2\Sigma_i\Sigma_jM_j\lambda_{i-j}k_i=\Sigma_iB_ik_i+g_{\lambda/k}\Sigma_ib_ik_i.$$

But for the terms with characteristic k we have

$$(2i+g_0)^2k_i-2\Sigma_iM_jk_{i-j}=0.$$

On substituting this in the previous equation, the left-hand member vanishes and we find

$$g_{\lambda/k}=-(\Sigma_iB_ik_i)\div(\Sigma_ib_ik_i) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

When this has been calculated, the equations may be solved by continued approximation. The existence of the relation (34) implies the arbitrariness of one of the λ_i: this has been defined to be such that $\lambda_0=0$ (§ 26).

The determination of g_0 has been made by ADAMS (*M. N.* vol. xxxviii., *Coll. Works*, vol. i.) and later by Mr. P. H. COWELL (*Amer. Jour. Math.* vol. xviii.) where full explanations of the method used will be found. The advance numerical results for g_{e^2}, $g_{e'^2}$, g_{k^2}, are given in the Appendix to Chap. IV. below, g_{k^2} having been found by Mr. COWELL in the paper just referred to.*

* Mr. Cowell objects to the above (which was given in a slightly different form in the *Investigations*) as a practical method, on account of the supposed length of the calculations. He, however, uses equation (8) to find g_{k^2} and the accompanying coefficients $(k^2k')_i$, while the above method contemplates the use of the equation (4). In the latter, the coefficients A_i or B_i, b_i are determined just as in any other set of inequalities; the only labour that remains in order to find $g_{\lambda/k}$ is the few minutes' work necessary to calculate the equation (34) above. The answer is, therefore, that the homogeneous equation (8), *or* the equation (4) should be used to calculate both the coefficients λ_i and $g_{\lambda/k}$, not one for the coefficients and the other for $g_{\lambda/k}$.

(d') *The case* $2i+\tau\pm g_0$ *small compared with unity. The inequalities whose periods are nearly equal to that of the principal term in latitude.* The remarks made in § 28 (d) apply also here with one or two evident changes; they need not, therefore, be repeated.

32. *The Homogeneous Equations* (6), (7) *for* u.—Suppose that it be required to determine the terms in $u\zeta^{-1}$ with characteristic λ and arguments $2i\pm\tau$, where τ is one of the values of $\pm pc\pm rm\pm 2qg$, those with lower characteristics having been found. It will be necessary here to slightly alter the notation of the last paragraphs by specifying the whole of the argument as well as the characteristic in the notation.

Let the particular terms in $u_\lambda\zeta^{-1}$ which have the arguments $2i\pm\tau$ be denoted by

$$a\lambda\Sigma_i(\lambda_{\tau,\,i}\zeta^{2i+\tau}+\lambda_{-\tau,\,i}\zeta^{2i-\tau}) \quad \dots \quad \dots \quad \dots \quad \dots \qquad (35)$$

The coefficients $\lambda_{\tau,i}$, $\lambda_{-\tau,i}$ are the unknowns, and are the same quantities as those previously denoted by λ_i, λ'_i.

The equations being of the second degree with respect to u, s, z, we must consider how terms with characteristic λ and arguments $2i\pm\tau$ arise in such expressions as $D^2(us)$, u^2, &c. The required terms will be made up of terms with characteristic μ, arguments $\pm(2i+\sigma)$, combined with terms with characteristic ν and arguments $\pm(2i+\tau-\sigma)$, where

$$\mu\nu=\lambda.$$

In conformity with the notation for terms with characteristic λ, let these terms be expressed by

$$\left.\begin{aligned} u_\mu\zeta^{-1}&=a\mu\Sigma_{\sigma,\,i}(\mu_{\sigma,\,i}\zeta^{2i+\sigma}+\mu_{-\sigma,\,i}\zeta^{2i-\sigma}) \\ u_\nu\zeta^{-1}&=a\nu\Sigma_{\sigma,\,i}(\nu_{\tau-\sigma,\,i}\zeta^{2i+\tau-\sigma}+\nu_{\sigma-\tau,\,i}\zeta^{2i-\tau+\sigma}) \end{aligned}\right\} \quad \dots \quad \dots \qquad (36)$$

It is evident that these may be made to include the terms (35) by putting $\mu=\lambda$, $\sigma=\tau$ in the first, or $\nu=\lambda$, $\sigma=0$ in the second.

To obtain the corresponding terms in s, we put $1/\zeta$ for ζ. As i receives negative as well as positive values, these may be written

$$\left.\begin{aligned} s_\mu\zeta&=a\mu\Sigma_{\sigma,\,i}(\mu_{-\sigma,\,-i}\zeta^{2i+\sigma}+\mu_{\sigma,\,-i}\zeta^{2i-\sigma}), \\ s_\nu\zeta&=a\nu\Sigma_{\sigma,\,i}(\nu_{\sigma-\tau,\,-i}\zeta^{2i+\tau-\sigma}+\nu_{\tau-\sigma,\,-i}\zeta^{2i-\tau+\sigma}), \end{aligned}\right\} \quad \dots \quad \dots \qquad (36')$$

Whence, for all terms with characteristic λ and arguments $2j+\tau$,

$$\left.\begin{aligned} us &= \mathrm{a}^2\lambda\Sigma_{\sigma,j,i}\mu_{\sigma,i}\nu_{\sigma-\tau,i-j}\zeta^{2j+\tau}, \\ D(u\zeta^{-1}) \,.\, D(s\zeta) &= \mathrm{a}^2\lambda\Sigma_{\sigma,j,i}(2i+\sigma)(2j-2i+\tau-\sigma)\mu_{\sigma,i}\nu_{\sigma-\tau,i-j}\zeta^{2j+\tau}, \\ u^2 &= \mathrm{a}^2\lambda\Sigma_{\sigma,j,i}\mu_{\sigma,i}\nu_{\tau-\sigma,j-i-1}\zeta^{2j+\tau}, \\ s^2 &= \mathrm{a}^2\lambda\Sigma_{\sigma,j,i}\mu_{-\sigma,i}\nu_{-\tau+\sigma,-j-i-1}\zeta^{2j+\tau}, \text{ etc.} \end{aligned}\right\} \quad \ldots \quad (37)$$

Since j, i were supposed to have the same range of values, it is evident that i may receive the values $0, \pm 1, \pm 2, \ldots$ while $2j$ receives *either* the values $0, \pm 2, \pm 4, \ldots$ *or* the values $\pm 1, \pm 3, \pm 5, \ldots$ according as λ contains even or odd powers of α. It is not necessary to specify summation with regard to μ or ν; it may be understood in the summation with regard to σ. The above expressions are so adjusted that the *same* set of values for σ will be available in all of them.

In general, for characteristics of orders which give sensible terms in the lunar theory, the number of values of μ, ν, σ is quite small. [For example, if $\lambda = \mathrm{e}^2e'$, $\tau = 2\mathrm{c}-\mathrm{m}$, we have the following pairs of values for μ, σ respectively: $1, 0$; e, c; e', $-\mathrm{m}$; e^2, $2\mathrm{c}$; $\mathrm{e}e'$, $\mathrm{c}-\mathrm{m}$; e^2e', $2\mathrm{c}-\mathrm{m}$. The corresponding values of ν are derived from the relation $\mu\nu = \lambda$.] When $\mu = 1$, $\sigma = 0$, the coefficient is $1_{0,i}$; it is that denoted previously by a_i.

The equations (6), (7) must now be put into the forms which will give results best adapted for numerical calculation when we substitute the expressions (37) in them.

The first process is the calculation of the terms involving z, ϖ_1 in the equations; they are evidently known terms, and the calculations consisting chiefly of multiplications, the latter do not call for special remark.

When these terms have been obtained the equation may be put into the form, after integrating (7),

$$L^2(us) - DuDs - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u+s)^2 = L \quad \ldots \quad \ldots \quad (38)$$

$$uDs - sDu - 2\mathrm{m}us + \frac{3}{2}\mathrm{m}^2D^{-1}(u^2 - s^2) = \Lambda \quad \ldots \quad \ldots \quad (39)$$

where L contains the known terms arising from z, ϖ_1, in (6) and $D\Lambda$ the known terms arising from ϖ_1 in (7). No arbitrary constant is needed in Λ, for the coefficient of ζ^0 always vanishes in (7).

The forms of the terms which go to make up L, Λ show immediately that, for terms with characteristic λ, and arguments $2i+\tau$, we shall have

$$L=a^2\lambda\Sigma_j L_j(\zeta^{2j+\tau}+\zeta^{-2j-\tau}),$$
$$\Lambda=a^2\lambda\Sigma_j\Lambda_j(\zeta^{2j+\tau}+\zeta^{-2j-\tau}).$$

The left-hand members of (38), (39) are also of similar form; hence, it is only necessary to equate the coefficients of $\zeta^{2j+\tau}$ to zero in (38), (39) in order to find those of $\zeta^{2j\pm\tau}$ in $u\zeta^{-1}$.

Multiply (39) by $2m+1$ and add to (38). The result may be written

$$D^2(us)-D(u\zeta^{-1})\,.\,D(s\zeta)-\left(1+2m-\frac{1}{2}m^2\right)us$$
$$+\frac{9}{4}m^2u^2+\frac{3}{2}m^2(2m+1)D^{-1}(u^2)+\frac{9}{4}m^2s^2-\frac{3}{2}m^2(2m+1)D^{-1}(s^2)$$
$$=L+(2m+1)\Lambda \qquad \ldots \qquad \ldots \qquad (40)$$

an equation which replaces (38).

33. Substituting the results (37) in (40), (39), and equating the coefficients of $\zeta^{2j+\tau}$ to zero, we find

$$\Sigma_{\sigma,i}\left[\left\{(2j+\tau)^2-(2i+\sigma)(2j-2i+\tau-\sigma)-1-2m+\frac{1}{2}m^2\right\}\mu_{\sigma,i}\nu_{\sigma-\tau,i-j}\right.$$
$$+\left(\frac{9}{4}m^2+\frac{3}{2}m^2\frac{2m+1}{2j+\tau}\right)\mu_{\sigma,i}\nu_{\tau-\sigma,j-i-1}$$
$$\left.+\left(\frac{9}{4}m^2-\frac{3}{2}m^2\frac{2m+1}{2j+\tau}\right)\mu_{-\sigma,i}\nu_{\sigma-\tau,-j-i-1}\right]=L_j+(2m+1)\Lambda_j \qquad \ldots \qquad \ldots \qquad (41)$$

$$\Sigma_{\sigma,i}\left[(2j-4i+\tau-2\sigma-2m-2)\mu_{\sigma,i}\nu_{\sigma-\tau,i-j}\right.$$
$$\left.+\frac{3}{2}\frac{m^2}{2j+\tau}\left\{\mu_{\sigma,i}\nu_{\tau-\sigma,j-i-1}-\mu_{-\sigma,i}\nu_{\sigma-\tau,-j-i-1}\right\}\right]=\Lambda_j \qquad \ldots \qquad \ldots \qquad \ldots \qquad (42)$$

The unknown quantities which are found by means of these equations are given by the values $\mu=\lambda$, $\sigma=\tau$, and $\mu=1$, $\sigma=0$. Since the a_i are known numerical quantities, the equations are linear with respect to the unknowns $\lambda_{\pm\tau,j}$ and they can be solved by the ordinary methods of continued approximation.

The equations written down are those of principal importance in finding $\lambda_{\tau,j}$, $\lambda_{-\tau,-j}$. The principal terms involving these two quantities are obtained by putting

$$\sigma=\tau,\ i=j\ ;\ \text{whence}\ \mu_{\sigma,i}=\lambda_{\tau,j}\ \text{and}\ \nu_{\sigma-\tau,i-j}=a_0=1,$$
$$\sigma=0,\ i=0\ ;\ \text{whence}\ \mu_{\sigma,i}=a_0=1\ \text{and}\ \nu_{\sigma-\tau,i-j}=\lambda_{-\tau,-j}.$$

The corresponding terms in the respective equations are

$$\left.\begin{array}{l}\left\{(2j+\tau)^2-1-2m+\frac{1}{2}m^2\right\}(\lambda_{\tau,j}+\lambda_{-\tau,-j}),\\ -(2j+\tau)(\lambda_{\tau,j}-\lambda_{-\tau,-j})-(2+2m)(\lambda_{\ ,j}+\lambda_{-\tau,-j}),\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

The method of solution of the simultaneous equations is therefore evident. We find the sum of the two unknowns from the first equation, and thence, substituting, their difference from the second equation.

The greater part of the labour of calculation consists in the computation of the first term of each equation, owing to the fact that the coefficients of $\mu_{\sigma,i}\nu_{\sigma-\tau,i-j}$ are different for different values of i, j. The coefficient in question in equation (41) is best written

$$\left(\tau^2-\sigma\tau+\sigma^2-1-2m+\frac{1}{2}m^2\right)+(4j^2-4ij+4i^2)+(4j-2i)\tau+(4i-2j)\sigma.$$

The first term of this remains the same while σ remains the same and when $\tau-\sigma$ is put for σ; the second term is always integral; the third and fourth terms require only multiplication by integers. Hence, after the first term has been obtained, no logarithmic multiplications are necessary to find the whole set of coefficients corresponding to the different values of i, j. The last remark applies also to the corresponding coefficient in (42). The rest of the numerical coefficients in both equations do not involve i.

34. We can deduce from (41), (42) forms in which the coefficient of $a_0\lambda_{\tau,j}$ is -1 and that of $a_0\lambda_{-\tau,-j}$ is 0, so that there will be no need to solve two simultaneous equations as the final step in each approximation to a pair of the unknowns. The details of the algebraical steps are sufficiently simple, and I merely give an outline.*

The expressions (43) show that if we multiply (41), (42) by

$$2m+2-2j-\tau, \qquad (2j+\tau)^2-1-2m+\frac{1}{2}m^2,$$

respectively, add and divide the resulting equation by

$$2(2j+\tau)\left\{(2j+\tau)^2-1-2m+\frac{1}{2}m^2\right\},$$

* Further details will be found in Chap. XI. (ii) of my *Treatise on the Lunar Theory*, where the spirit of this method is applied to a particular case.

the coefficient of $\lambda_{-\tau,-j}$ will be zero, and that of $\lambda_{\tau,j}$ will be -1. The result is

$$\Sigma_{\sigma,i}\Big[[2j+\tau,\ 2i+\sigma]\mu_{\sigma,i}\nu_{\sigma-\tau,\,i-j}+[2j+\tau,]\mu_{\sigma,i}\nu_{\tau-\sigma,\,-i-1} \\ +(2j+\tau,)\mu_{-\sigma,i}\nu_{\sigma-\tau,\,-j-i-1}\Big]=H_{\tau,j} \quad \dots \quad \dots \quad \dots \quad \dots \quad 44)$$

where

$$\left.\begin{aligned} [2j+\tau,\ 2i+\sigma] &= -\frac{2i+\sigma}{2j+\tau}\ \frac{2(2j+\tau)^2-2-4\mathrm{m}+\mathrm{m}^2+(2i+\sigma-2j-\tau)(2j+\tau-2-2\mathrm{m})}{2(2j+\tau)^2-2-4\mathrm{m}+\mathrm{m}^2} \\ [2j+\tau,] &= -\frac{3}{4}\mathrm{m}^2\frac{(2j+\tau)^2-4(2j+\tau)-2-2(2j+\tau+4)\mathrm{m}-9\mathrm{m}^2}{(2j+\tau)^2\{2(2j+\tau)^2-2-4\mathrm{m}+\mathrm{m}^2\}} \\ (2j+\tau,) &= -\frac{3}{4}\mathrm{m}^2\frac{5(2j+\tau)^2-8(2j+\tau)+2-2(10j+5\tau-4)\mathrm{m}+9\mathrm{m}^2}{(2j+\tau)^2\{2(2j+\tau)^2-2-4\mathrm{m}+\mathrm{m}^2\}} \end{aligned}\right\} \quad \dots \quad (45)$$

$$H_{\tau,j}=\frac{(2\mathrm{m}+2-2j-\tau)L_j+\left\{(2j+\tau)^2-1-2\mathrm{m}+\frac{1}{2}\mathrm{m}^2+(2\mathrm{m}+1)(2\mathrm{m}+2-2j-\tau)\right\}\Lambda_j}{(2j+\tau)\{2(2j+\tau)^2-2-4\mathrm{m}+\mathrm{m}^2\}} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (46)$$

The equation for $\lambda_{-\tau,-j}$ is obtained from (44) by changing the signs of τ, j.

35. The case (c) of § 28 deserves mention in connection with equation (44). It will be noticed that whenever the *square* of $2i+\tau$ appears as a divisor, namely, in the second and third terms of (44) and in $\Lambda_j/(2j+\tau)$, (since Λ_j already contains it as a divisor), the terms have the factor m^2. The use of equation (7) or (42) for the calculation of the coefficients of long-period inequalities has been noticed in § 28 (c). Thus (41), (42) and (44) are free from the objection remarked in that paragraph.

It may be also noticed that the expressions (43) show that when $2j+\tau$ is small, the corresponding loss of accuracy in the results arises in the *difference* of $\lambda_{\tau,j}$, $\lambda_{-\tau,-j}$ and not in their *sum*. This, translated to polar coordinates, is equivalent to saying that the loss of accuracy in the coefficients of long-period inequalities is chiefly felt in the longitude and but little in the parallax. In the case of short-period inequalities with small divisors the loss of accuracy falls on both coordinates. See also § 31.

With regard to the relative advantages and disadvantages of the equations (3), (4) and the homogeneous equations, from the point of view of actual calculation, it seems, from the experience gained in using both forms, that on the whole the advantage lies with the former. This is certainly true for inequalities of the second order and very probably for those of the third order

also. For inequalities of higher orders, the operations in the former are numerous but simple, and capable of continual verification; those in the latter are rather less numerous but more complicated, more productive of error, less easily verified at the various stages, and are much less easily arranged for a computer. The equations (3), (4) are now being used for the inequalities of the third order.

If the homogeneous equations be used, the expansions (41), (42) are preferable in actual calculation to (44). This remark does not, of course, apply to the inequalities of zero order, computed by Dr. HILL in the *Researches*.

36. *The Homogeneous Equation* (8) *or* (8′) *for* z may be treated in a similar manner, and a formula similar to (41) or (42) obtained for it. The calculations and results are much simpler, first on account of the less complicated form of (8); and, secondly, on account of the fact that there is only *one* equation of principal importance for the determination of each unknown, instead of two equations for each pair of unknowns. As the equation (4) will probably be used for all inequalities, I shall not develope the equation (8).

The motions of the perigee and node are found by continued approximation along with the unknown coefficients. It is possible to eliminate all the latter and to get single equations for the parts of the motions of the perigee or node corresponding to those given in §§ 28, 31 above; but this will be no saving of time, as the equations are somewhat complicated. The formulæ necessary will therefore not be given here.

Section (viii). *Calculation and Verification.*

37. It has already been stated that one of the chief objects aimed at in the developments given above was the reduction of the calculations to forms which made them as far as possible merely mechanical. This is the case with the right-hand members of (17), (18) or (17′), (18′). In the case of the former, $\frac{\kappa u_0 \zeta^{-1}}{\mathrm{a}\rho_0{}^3}$, $\frac{\kappa}{\mathrm{a}\rho_0{}^2}$, are calculated, once for all, from the values of u_0, s_0. At each stage of the approximations u_μ/u_0 is obtained from the value of u_μ by multiplication of $u_\mu\zeta^{-1}$ by $1/u_0\zeta^{-1} = s_0\zeta/\rho_0{}^2$, the last quantity being found by "special values." The remainder of the calculations of the right-hand members are then simple multiplications of ζ series. In the case of the equations (17′), (18′) (which were used for the results of Chap. IV. and for some of

those which will be published in Chap. V.) P, $\bar{P}$, Q, . . . were calculated by the method of special values from those of u_0, s_0; the remainder of the process is then as before.

38. The plan adopted for the multiplication of any two ζ-series, say

$$(\Sigma_i \alpha_i \zeta^{2i+\alpha})(\Sigma_i \beta_i \zeta^{2i+\beta}) = \Sigma_{i,j} \alpha_{j-i} \beta_i \zeta^{2j+\quad\beta}$$

consists in taking out the logarithms of the α_i and arranging them along a slip of paper in the order . . . α_2, α_1, α_0, α_{-1}, α_{-2}, . . . ; the logarithms of the β_i are arranged along another slip in the order . . . β_{-2}, β_{-1}, β_0, β_1, β_2. . . . The two slips being placed over one another, the sums of all the logarithms for a given value of j are taken without moving the slips, and they are written down in a column. The number corresponding to each logarithm is then taken from the tables and the results added for each value of j. Thus to find the coefficient of $\zeta^{4+\alpha+\beta}$ in the product, that is $\Sigma_i\ \alpha_{2-i}\ \beta_i$, the slips are placed so that α_2 falls under β_0, then α_1 falls under β_1, &c. The arrangement of this part of the sheet is then—

Values of j	i	log $\alpha_{j-i}\beta_i$	number
2	..	..	..
	-2	..	..
	-1	..	..
	0	..	..
	..	..	..
Sum, $j=2$			..

The process can be thus arranged for a computer, and the mere copying of figures from one sheet to another is very rarely necessary.

The result of each multiplication of series is verified by adding the sums for all values of j. The sum should be equal to $(\Sigma_i\alpha_i)(\Sigma_i\beta_i)$.

39. The values of the A_i, A'_{-i} in equations (26), (32) are in general carried to the same number of places of decimals for each value of i. In the solution of the equations of condition, large divisors frequently occur for large values of i and the λ_i, λ'_i are obtained to one or two more places of decimals for large values of i. Thus it is in general possible to find D^2u_λ to the same degree of accuracy as the A_i, A'_i. Exception only occurs in the cases of some long and short period inequalities; when it occurs, the corresponding values of

A_i, A'_i are taken to one or two more places of decimals or the homogeneous equation (7) is used, as explained in § 28 (*c*).

The object of taking the values of λ_i, λ'_i to more decimals for large values of i is to render the equation of verification more searching. For verification I use one of the homogeneous equations with $\zeta=1$; the calculation of it is never very long, and it appears to furnish a good test. See *Investigations*, p. 343.

40. In the following chapters I give in general two sets of numerical results: First, the values of the right-hand coefficients A_i, A'_i; and, secondly, the values of λ_i, λ'_i. They are taken exactly as they stood in my manuscripts. Although many of them will not be more than two units wrong in the last place given, the number of the calculations prevents this being said of all. They are intended to be trustworthy as far as the last figure but one in each case. The sums of the numbers in each column are always given, so that any error of transcription or typography may be detected should it occur.

All calculations are made at least twice, separated by an interval of time. The general plan, when several hours a day were available, was to have two or three separate sets of calculations proceeding together. Each of these would be carried to a certain stage and, after the lapse of a day or two, they would be gone over again, the errors, if any, corrected, the results verified (whenever this was possible) and they would then be taken a stage further. In this way an error running through several pages of calculations was avoided.

Section (ix). *Transformation to Polar Coordinates.*

41. We have, by equations (9), Section (iv),

$$\rho \exp.(V-nt-\epsilon)\iota=u\zeta^{-1},$$
$$\rho \exp.-(V-nt-\epsilon)\iota=s\zeta,$$
$$\rho \tan \phi=z.$$

Hence

$$2\iota(V-nt-\epsilon)=\log u\zeta^{-1}-\log s\zeta,$$
$$\frac{1}{r}=\frac{1}{(us+z^2)^{\frac{1}{2}}}$$
$$\phi=\tan^{-1}\frac{z}{\rho}.$$

Let V_0 be the part of V corresponding to the values u_0, s_0 of u, s and let

$$\begin{aligned} V-nt-\epsilon&=V_0+\Sigma V_\mu, \\ u&=u_0+\Sigma u_\mu, \\ s&=s_0+\Sigma s_\mu, \\ z&=\Sigma z_\mu. \end{aligned}$$

Then

$$2\iota V_0=\log u_0\zeta^{-1}-\log s_0\zeta,$$

$$2\iota(V_0+\Sigma V_\mu)=\log (u_0+\Sigma u_\mu)\zeta^{-1}-\log (s_0+\Sigma s_\mu)\zeta,$$

and

$$\begin{aligned} 2\iota\Sigma V_\mu&=\log\left(1+\frac{\Sigma u_\mu}{u_0}\right)-\log\left(1+\frac{\Sigma s_\mu}{s_0}\right) \\ &=\frac{\Sigma u_\mu}{u_0}-\frac{\Sigma s_\mu}{s_0}-\frac{1}{2}\left(\frac{\Sigma u_\mu}{u_0}\right)^2+\frac{1}{2}\left(\frac{\Sigma s}{s_0}\right)^2+\quad\dots\quad\dots\quad\dots \qquad (47) \end{aligned}$$

Also

$$\tan V_0=\frac{1}{\iota}\frac{u_0\zeta^{-1}-s_0\zeta}{u_0\zeta^{-1}+s_0\zeta}=\frac{\Sigma_i(a_i-a_{-i})\sin 2iD}{\Sigma_i(a_i+a_{-i})\cos 2iD}\quad\dots\quad\dots\quad\dots \qquad (48)$$

The value of V_0 may be calculated by the method of special values from those of a_i. The various terms of the right-hand member of (47) will have been already found in the calculation of the inequalities. Whence by addition we obtain the true longitude V.

42. With the same substitutions, we have

$$\frac{a}{r}=\frac{a}{r_0}\left[1+\frac{\Sigma u_\mu}{u_0}+\frac{\Sigma s_\mu}{s_0}+\frac{\Sigma u_\mu}{u_0}\cdot\frac{\Sigma s_\mu}{s_0}+\left(\frac{\Sigma z_\mu}{\rho_0}\right)^2\right]^{-\frac{1}{2}}\dots\quad\dots\quad\dots \qquad (49)$$

where

$$n^2a^3=E+M.$$

The right-hand member of this equation is then expanded. The various products will have been found, as before. All that remains to find a/r is to multiply the result by

$$\frac{a}{r_0}=\frac{a}{\mathrm{a}}\cdot\frac{\mathrm{a}}{\rho_0} \qquad (\text{since } r_0=\rho_0),$$

which is found from Chap. II.

We may also use equation (5) of § 7 to find $1/r$, the eonstant term being obtained by (49) or by the method eontained in Part ii. of the *Investigations*.

43. Finally

$$\phi=\frac{z}{\rho}-\frac{1}{3}\left(\frac{z}{\rho}\right)^3+\frac{1}{5}\left(\frac{z}{\rho}\right)^5-\ldots\ldots,$$

and

$$\frac{z}{\rho}=\frac{\Sigma z_\mu}{\rho_0}\left[1+\frac{\Sigma u_\mu}{u_0}+\frac{\Sigma s_\mu}{s_0}+\frac{\Sigma u_\mu}{u_0}\cdot\frac{\Sigma s_\mu}{s_0}\right]^{-\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

The expansions and multiplications will have all been performed, and thence ϕ may be easily found.

Henee the whole proeess of transforming to polar eoordinates will be first, for the longitude, the addition of eertain known series and the ealeulation of V_0; seeondly, for the parallax, the addition of known series and a multiplieation of the whole by a/ρ_0; thirdly, for the latitude, the addition of known series.

CHAPTER II

Terms of Order Zero.

Seetion (i). *Values of a_i,* a.

44. The coefficients of order zero have been obtained by Dr. Hill * to 15 places of deeimals. They are given by the particular solution of equations (12) or (12′) of § 18. This solution is expressed by

$$u_0\zeta^{-1}=\mathrm{a}\Sigma_i a_i\zeta^{2i}, \qquad \text{where } a_0=1.$$

The value of $\mathrm{m}=n'/(n-n')$ used is

$$\mathrm{m}=+\cdot 08084 \quad 89338 \quad 08311 \quad 6.$$

* "Researches in the Lunar Theory," *Amer. Jour. Math.*, vol. i. pp. 247–249. The coefficient denoted above by $\mathrm{a}a_i$ is denoted by a_i in Dr. Hill's paper.

Values of

i.	a_i.
6	$+ \cdot 00000 \quad 00000 \quad 00007$
5	$+ \cdot 00000 \quad 00000 \quad 01107$
4	$+ \cdot 00000 \quad 00001 \quad 75268$
3	$+ \cdot 00000 \quad 00300 \quad 31632$
2	$+ \cdot 00000 \quad 58786 \quad 56578$
1	$+ \cdot 00151 \quad 57074 \quad 79563$
0	$+1$
-1	$- \cdot 00869 \quad 57469 \quad 61540$
-2	$+ \cdot 00000 \quad 01637 \quad 90486$
-3	$+ \cdot 00000 \quad 00024 \quad 60393$
-4	$+ \cdot 00000 \quad 00000 \quad 12284$
-5	$+ \cdot 00000 \quad 00000 \quad 00064$
-6	$+ \cdot 00000 \quad 00000 \quad 00000$
Sum ...	$+ \cdot 99282 \quad 60356 \quad 45842$

The relation between a and a where

$$n^2a^3 = E + M,$$

is given by

$$\frac{\mathrm{a}}{a} = + \cdot 99909 \quad 31419 \quad 75298.$$

This relation will not be required after the end of this Chapter until we come to the deduction of the lunar parallax and the expression of the coefficients containing $\alpha = \mathrm{a}/a'$ in terms of a/a'.

Section (ii).—*Values of M, N, P, etc.*

45. From the results of the previous section we deduce the series for

$$M,\ N,\ P,\ Q,\ R,\ S,\ T, \qquad \frac{\mathrm{a}}{u_0\zeta^{-1}},\ \frac{\mathrm{a}}{\rho_0},\ \frac{\kappa u_0\zeta^{-1}}{\mathrm{a}\rho_0^{\ 3}},\ \frac{\kappa}{\mathrm{a}\rho_0^{\ 2}}.$$

See Chap. I., §§ 19, 20, 37. The series for $\bar{P}$, $\bar{Q}$, . . . are obtained from those for P, Q, . . . by putting $1/\zeta$ for ζ; this is the same as putting $-i$ for i in the suffixes of the coefficients.

$M=\Sigma_i M_i \zeta^{2i}, \qquad N=\Sigma_i N_i \zeta^{2i}.$

Values of

i.	M_i.	N_i.
5	+·00000 00000 5	+ ·00000 00009 0
4	+·00000 00056 5	+ ·00000 00824 6
3	+·00000 06029 7	+ ·00000 70129 7
2	+·00006 28883 4	+ ·00054 79401 6
1	+·00630 84231 2	+ ·03686 55171 8
0	+·58902 22856 4	+1·75707 88032 7
−1	+·00630 84231 2	+ ·01078 63527 2
−2	+·00006 28883 4	+ ·00001 25690 4
−3	+·00000 06029 7	+ ·00000 00982 3
−4	+·00000 00056 5	+ ·00000 00007 6
−5	+·00000 00000 5	
Sum ...	+·60176 61259 0	+1·80529 83776 9

The series M is given by Dr. Hill in his paper just referred to. The series M, N are both given on p. 328 of the *Investigations*.

$P=\Sigma P_i \zeta^{2i}, \qquad \overline{P}=\Sigma P_{-i} \zeta^{2i}, \qquad Q=\Sigma Q_i \zeta^{2i}.$*

Values of

i.	P_i.	Q_i.
5	+ ·00000 00005	+ ·00000 00020
4	+ ·00000 00465	+ ·00000 01462
3	+ ·00000 40164	+ ·00001 04704
2	+ ·00032 38766	+ ·00066 73632
1	+ ·02280 40093	+ ·03476 15314
0	+1·17156 77322	+1·17132 34260
−1	+ ·01084 18484	− ·00112 12092
−2	+ ·00010 24640	+ ·00000 31923
−3	+ ·00000 09526	+ ·00000 00327
−4	+ ·00000 00092	+ ·00000 00007
−5	+ ·00000 00001	
Sum ...	+1·20564 49558	+1·20564 49557

* The series $\overline{Q}$ will not be required.

$$R=\Sigma R_i\zeta^{2i},\quad S=\Sigma S_i\zeta^{2i},\quad T=\Sigma T_i\zeta^{2i}$$
$$\bar{R}=\Sigma R_{-i}\zeta^{2i},\quad \bar{S}=S\quad \bar{T}=\Sigma T_{-i}\zeta^{2i}$$

Values of

i.	R_i.	S_i.	T_i.
5	+ ·00000 00016	+ ·00000 00004	+ ·00000 00047
4	+ ·00000 01268	+ ·00000 00384	+ ·00000 03170
3	+ ·00000 93162	+ ·00000 34249	+ ·00001 96595
2	+ ·00061 05520	+ ·00028 51417	+ ·00105 79688
1	+ ·03299 09904	+ ·02103 03831	+ ·04494 46487
0	+ 1·17159 65655	+ 1·17171 87304	+ 1·17123 01076
−1	+ ·00906 65690	+ ·02103 03831	− ·00289 67092
−2	+ ·00008 18396	+ ·00028 51417	+ ·00000 06964
−3	+ ·00000 07394	+ ·00000 34249	+ ·00000 00136
−4	+ ·00000 00066	+ ·00000 00384	+ ·00000 00006
−5	+ ·00000 00001	+ ·00000 00004	
Sum ...	+ 1·21435 67072	+ 1·21435 67074	+ 1·21435 67077

The series P, Q are given on p. 328 of the *Investigations*. The coefficient R_i of that memoir is here called R_{-i}. The coefficients Q_3, Q_{-3} are corrected here, each of them having been diminished by one unit in the ninth place.

46. Values of the coefficients of ζ^{2i} in the expansions of

i.	$\frac{a}{u_0\zeta^{-1}}$.	$\frac{a}{\rho_0}$.
5		+ ·00000 00000 1
4		+ ·00000 00012 2
3	− ·00000 00156	+ ·00000 01632 9
2	− ·00000 35815	+ ·00002 29007 0
1	− ·00151 57497	+ ·00358 99818 9
0	+ ·99997 36392	+ ·99999 97077 7
−1	+ ·00869 54035	+ ·00358 99818 9
−2	+ ·00007 54483	+ ·00002 29007 0
−3	+ ·00000 06521	+ ·00000 01632 9
−4	+ ·00000 00056	+ ·00000 00012 2
−5		+ ·00000 00000 1
Sum ...	+ 1·00722 58019	+ 1·00722 58019 9

Values of the coefficients of ζ^{2i} in the expansions of

i.	$\frac{\kappa u_0 \zeta^{-1}}{a\rho_0^{3}}$.	$\frac{\kappa}{a\rho_0^{2}}$.
5	+ ·00000 00001 3	+ ·00000 00000 4
4	+ ·00000 00144 3	+ ·00000 00048 5
3	+ ·00000 15058 3	+ ·00000 05751 9
2	+ ·00015 17769 1	+ ·00006 87511 8
1	+ ·01439 14901 8	+ ·00841 09322 7
0	+1·17141 74324 9	+1·17144 79701 8
−1	+ ·00242 99016 5	+ ·00841 09322 7
−2	+ ·00001 62575 2	+ ·00006 87511 8
−3	+ ·00000 01171 8	+ ·00000 05751 9
−4	+ ·00000 00008 8	+ ·00000 00048 5
−5	+ ·00000 00000 1	+ ·00000 00000 4
Sum ...	+1·18840 84972 1	+1·18840 84972 4

These four series will not be required until Chap. V., as the calculations of Chap. IV. were made with P, $\overline{P}$, Q; for a few of the inequalities of Chap. V. the series R, $\overline{R}$, S, T have been used.

CHAPTER III

TERMS OF THE FIRST ORDER.

47. The terms of the first order have been treated in Chap. I., § 19. The results contained in this chapter are classified in the following table :—

Section.	Characteristic.	Arguments.	Quantities found here.	
			Coefficients.	Motions of Args.
(i)	e	$2i \pm c$	ϵ_i, ϵ'_i	c_0
(ii)	e'	$2i \pm m$	η_i, η'_i	
(iii)	a	$2i_1$	$(a)_{i_1}$	
(iv)	k	$\pm(2i+g)$	k_i, $k'_{-i}(=-k_i)$	g_0

where

$$i=0, \pm 1, \pm 2, \ldots, 2i_1, = \pm 1, \pm 3, \pm 5 \ldots$$

The formulæ furnished by equations (15), (16) of Chap. I. will be given in each case, although the results may have been otherwise obtained. References will be made to all previously published results.

Section (i).—*Characteristic* e. *Value of* c_0.

48. The value of c_0, which is the part of c depending only on m, has been found by Dr. HILL * to 15 places of decimals. It is,

$$c_0 = +1{\cdot}07158\ \ 32774\ \ 16012.$$

The equation satisfied by c_0 and the terms with characteristic e is

$$\zeta^{-1}(D+m)^2 u_e + M u_e \zeta^{-1} + N s_e \zeta = 0.$$

The solution is

$$u_e \zeta^{-1} = ae\Sigma_i(\epsilon_i \zeta^{2i+c} + \epsilon'_i \zeta^{2i-c}).$$

The equations of condition for the unknowns having been obtained by substituting the assumed solution in the differential equation, and equating the coefficients of the various powers of ζ to zero, we may solve them with the above value of c, so as to give ϵ_i, ϵ'_i in terms of ϵ_0, ϵ'_0.

Let

$$\epsilon_i = b_i \epsilon_0 + \beta_i \epsilon'_0, \qquad \epsilon'_i = b'_i \epsilon_0 + \beta'_i \epsilon'_0.$$

Values of

i.	b_i.	β_i.
5		
4	+ ·00000 00005	− ·00000 00006
3	+ ·00000 00843	− ·00000 00708
2	+ ·00001 47376	− ·00000 85378
1	+ ·00308 02927	− ·00092 80067
0	+1	0
−1	+ ·01999 88763	+ ·20567 90112
−2	+ ·00001 15205	+ ·00007 34691
−3	− ·00000 00193	− ·00000 01734
−4	− ·00000 00001	− ·00000 00012
−5		
Sum ...	+1·02310 54925	+ ·20481 56898

* "Motion of the Perigee, etc.," *Acta Math.* vol. viii. p. 35.

Values of

i.	b_i'.	β_i'.
5		− ·00000 00002
4	− ·00000 00029	− ·00000 00212
3	− ·00000 04039	− ·00000 29218
2	− ·00005 93876	− ·00043 20782
1	− ·01054 68058	− ·07779 55430
0	0	+ 1
−1	− ·00108 65960	− ·00019 59999
−2	+ ·00000 01043	− ·00000 08618
−3	+ ·00000 00024	− ·00000 00055
−4		
−5		
Sum ...	− ·01169 30895	+ ·92157 25684

The arbitrary constant e is defined (Chap. I., § 25) to be such that

$$\epsilon_0 - \epsilon_0' = 1.$$

Either of the two remaining equations of condition (those of principal importance for finding ϵ_0, ϵ'_0) then gives

$$\epsilon_0 + \epsilon'_0 = -\cdot 49679 \quad 18022.$$

From these two equations we find ϵ_0, ϵ'_0, and thence, from the numbers just given, the values of ϵ_i, ϵ'_i.

Values of

i.	ϵ_i.	ϵ_i'.
5		+ ·00000 00001
4	+ ·00000 00005	+ ·00000 00152
3	+ ·00000 00742	+ ·00000 20851
2	+ ·00001 00977	+ ·00030 84234
1	+ ·00146 95307	+ ·05556 82459
0	+ ·25160 40989	− ·74839 59011
−1	− ·14889 75297	− ·00012 67065
−2	− ·00005 20854	+ ·00000 06713
−3	+ ·00000 01250	+ ·00000 00048
−4	+ ·00000 00009	
−5		
Sum ...	+ ·10413 43128	− ·69264 31618

I obtained these results by the use of the homogeneous equations.* A different set of values for b_i, b'_i, β_i, β'_i will naturally arise if we use the equation at the beginning of this section. One slight error which occurred in the reduction of $b'_4\epsilon_0+\beta'_4\epsilon_0$ to the final value of ϵ'_4 was discovered and corrected.

The short-period inequality with a small divisor is the "Evection"; the corresponding coefficients are ϵ_{-1}, ϵ'_1.

Section (ii).—*Characteristic e'.*

49. The equation is

$$\zeta^{-1}(D+\mathrm{m})^2u_{e'}+Mu_{e'}\zeta^{-1}+Ns_{e'}\zeta=-\frac{\partial\Omega_1}{\partial s}\zeta^{-1}.$$

In the right-hand member we put $\Omega_1=\omega_2$, $z=0$, $u=u_0$, $s=s_0$, and neglect powers of e' above the first (Chap. I., § 19).

Hence, by Chap. I., Sect. (iii),

$$\frac{\partial\Omega_1}{\partial s}\zeta^{-1}=\mathrm{m}^2\left[\frac{3}{2}\mathfrak{a}_2s_0+\frac{1}{2}\mathfrak{b}_2u_0\right]\zeta^{-1}$$

$$=\frac{3}{4}\mathrm{m}^2e'[(u_0\zeta^{-1}+7s_0\zeta\,.\,\zeta^{-2})\zeta^{\mathrm{m}}+(u_0\zeta^{-1}-s_0\zeta\,.\,\zeta^{-2})\zeta^{-\mathrm{m}}].$$

The solution is

$$u_{e'}\zeta^{-1}=\mathrm{a}e'\Sigma_i(\eta_i\zeta^{2i+\mathrm{m}}+\eta'_i\zeta^{2i-\mathrm{m}}).$$

The equations of condition are formed and then solved by continued approximation.

Values of

i.	η_i.	η_i'.
5	$-$·00000 00000 03	$+$·00000 00000 24
4	$-$·00000 00004 40	$+$·00000 00030 59
3	$-$·00000 00572 63	$+$·00000 03956 99
2	$-$·00000 76025 41	$+$·00005 22794 42
1	$-$·00103 48418 2	$+$·00695 08210 5
0	$-$·09186 93227	$+$·09869 89451
-1	$-$·03636 42746 8	$+$·00448 82585 5
-2	$+$·00000 17438 21	$-$·00000 01475 00
-3	$+$·00000 00322 23	$-$·00000 00041 96
-4	$+$·00000 00002 08	$-$·00000 00000 29
-5	$+$·00000 00000 01	
Sum ...	$-$·12927 43232	$+$·11019 05512

* "The Elliptic Inequalities in the Lunar Theory," *Amer. Jour. Math.* vol. xv. pp. 259–261.

The above method was used to calculate all these coefficients. The long period inequality with a small divisor is the "Annual Equation," having the coefficients η_0, η'_0. The method of § 29 was used in the approximations to these two coefficients.

The values of the corresponding terms in the true longitude have been given in a note in the *Monthly Notices*, vol. liv. p. 471.

Section (iii). *Characteristic* $\alpha = a/a'$.

50. The equation is

$$\zeta^{-1}(D+m)^2 u_\alpha + M u_\alpha \zeta^{-1} + N s_\alpha \zeta = -\frac{\partial \Omega_1}{\partial s}\zeta^{-1}.$$

In the right-hand member we put $\Omega_1 = \omega_3$, $z=0$, $e'=0$, $u=u_0$, $s=s_0$. Hence, by Chap. I., Sect. (iii),

$$\frac{\partial \Omega_1}{\partial s}\zeta^{-1} = a\alpha \cdot \frac{3}{4}m^2 \cdot \frac{1}{a^2}\left[\frac{5}{2}(s_0\zeta)^2\zeta^{-3} + \frac{1}{2}(u_0\zeta^{-1})^2\zeta + (u_0 s_0)\zeta^{-1}\right].$$

The solution is

$$u_\alpha \zeta^{-1} = a\alpha \Sigma_i (\alpha)_i \zeta^{2i},$$

where

$$2i = \pm 1, \ \pm 3, \ \pm 5 \ \ldots .$$

Values of

$2i$.	$(\alpha)_i$.
9	$+\cdot 00000 \ \ 00001$
7	$+\cdot 00000 \ \ 00072$
5	$+\cdot 00000 \ \ 04839$
3	$-\cdot 00005 \ \ 88448$
1	$-\cdot 06417 \ \ 03547$
-1	$+\cdot 17899 \ \ 19628$
-3	$-\cdot 00293 \ \ 82096$
-5	$-\cdot 00000 \ \ 18325$
-7	$-\cdot 00000 \ \ 00029$
-9	
Sum ...	$+\cdot 11182 \ \ 32095$

These coefficients I found to seven places of decimals in a paper "On the Parallactic Inequalities in the Lunar Theory"* by the use of the homogeneous equations. They have been recalculated and extended to ten places by the above method; errors of one unit only in the sixth places of decimals in the values of $a_{-\frac{1}{2}}$, $a_{\frac{3}{2}}$ were detected. Dr. HILL in his paper "On the Periodic Solution, &c.,"† using my former values as a first approximation has also recalculated these terms to a high degree of accuracy by a totally different method.

The short-period inequality with a small divisor is the "Parallactic Inequality," having the coefficients $a_{\frac{1}{2}}$, $a_{-\frac{1}{2}}$.

Section (iv). *Characteristic* k. *Value of* g_0.

51. The part of the value of g which depends on m only, namely g_0, has been obtained by Professor J. C. ADAMS and Mr. P. H. COWELL (see the references in § 31). The latter finds

$$g_0 = 1{\cdot}08517\ \ 14265\ \ 58.$$

The slightly different result obtained by ADAMS is due to the use of a different value for m.

The equation giving g_0 and the terms with characteristic k is

$$D^2 z_k - 2Mz_k = 0.$$

The solution is

$$z_k\ \iota = \mathrm{ak}\Sigma_i\ k_i\ (\zeta^{2i+g} - \zeta^{-2i-g}).$$

The constant k is defined (Chap. I., § 26) to be such that

$$k_0 = 1.$$

* *Amer. Jour. Math.* vol. xiv. p. 157. A different notation is there used.

† *Astron. Jour.* vol. xv. pp. 137–143. Dr. HILL informs me that the large correction which he obtained to my value of the coefficient of the Parallactic Inequality n longitude, amounting to 5 units in the fifth place of decimals in the value of $a_{\frac{1}{2}} - a_{-\frac{1}{2}}$, was due to a slight error in reducing them to his form. As the resulting value was only used as a first approximation, his final results are, of course, correct.

Mr. COWELL finds the following values (*loc. cit.* p. 113).

Values of

i.	k_i.
5	+ ·00000 00000 01
4	+ ·00000 00001 75
3	+ ·00000 00299 82
2	+ ·00000 58673 61
1	+ ·00151 22192 28
0	+ 1
−1	− ·03698 39313 94
−2	− ·00004 65750 01
−3	− ·00000 01755 37
−4	− ·00000 00008 87
−5	− ·00000 00000 05
Sum ...	+ ·96448 74339 23

I have verified these results by means of the homogeneous equation (7) with $\aleph_1=0$, by putting $\zeta=+1, -1$, successively, after the substitution of the values in the equation.

The short-period term with a small divisor is that having the numerical coefficient k_{-1}.

CHAPTER IV

TERMS OF THE SECOND ORDER.

Section (i). *Formulæ.*

52. The general type of the equation for u for terms with characteristic λ, and arguments $2i\pm\tau$, is, by Chap. I., § 20,

$$\zeta^{-1}(D+\mathrm{m})^2\, u_\lambda + M u_\lambda\, \zeta^{-1} + N s_\lambda \zeta = \mathrm{a}\lambda A \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where

$$A=\Sigma_i\,(A_i\,\zeta^{2i+\tau}+A'_i\,\zeta^{2i-\tau}) \text{ when } \tau\neq 0,$$

and

$$A=\Sigma_i\,A_i\,\zeta^{2i} \text{ when } \tau=0.$$

Here, by Chap. I., equation (17′),

$$\mathrm{a}\lambda A = \text{Part, char}^{\text{c}}.\ \lambda, \text{ in } \frac{1}{\mathrm{a}}\left[-\mathrm{a}\frac{\partial \varpi_1}{\partial s}\zeta^{-1}\right.$$

$$\left.+\frac{3}{8}\bar{P}\,(\Sigma u_\mu\,\zeta^{-1})^2+\frac{15}{8}\,Q\,(\Sigma s_\mu\,\zeta)^2+\frac{3}{4}P\,(\Sigma u_\mu)\,(\Sigma s_\mu)-\frac{3}{2}P\,(\Sigma z_\mu)^2\right] \quad \ldots \quad (2)$$

$$\Sigma u_\mu = u_{\mathrm{e}} + u_{e'} + u_a, \qquad \Sigma z_\mu = z_{\mathrm{k}}.$$

In all cases s is derived from u by putting $1/\zeta$ for ζ. The first term of (17′) contributes nothing to the terms of the second order.

Also, by Chap. I., Sect. (iii), substituting for u, s, z and neglecting powers and products of u_μ, s_μ, z_μ, we have

$$\frac{\partial \varpi_1}{\partial s} = \mathrm{m}^2\left[\frac{3}{2}\,(s_0+\Sigma s_\mu)\,\bar{\mathfrak{a}}_2+\frac{1}{2}\,(u_0+\Sigma u_\mu)\,\mathfrak{b}_2\right]$$

$$+\frac{\mathrm{m}^2}{a'}\left[\frac{15}{8}\,(s_0^2+2\,s_0\,\Sigma s_\mu)\,\bar{\mathfrak{a}}_3+\frac{3}{8}\,(u_0^2+2\,u_0\,\Sigma u_\mu)\,\mathfrak{c}_3+\frac{3}{4}\,(u_0\,s_0+u_0\,\Sigma s_\mu+s_0\,\Sigma u_\mu)\bar{\mathfrak{c}}_3\right]$$

$$+\frac{\mathrm{m}^2}{a'^2}\left[\frac{35}{16}\,s_0^3+\frac{5}{16}\,u_0^3+\frac{15}{16}\,u_0\,s_0^2+\frac{9}{16}\,u_0^2\,s_0\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where

$\bar{\mathfrak{a}}_2$, $\mathfrak{b}_2$ take their values as far as e'^2 when multiplied by u_0, s_0 ;

$\bar{\mathfrak{a}}_2$, $\mathfrak{b}_2$ take their values as far as e' when multiplied by u_μ, s_μ ;

$\bar{\mathfrak{a}}_3$, $\mathfrak{c}_3$, $\bar{\mathfrak{c}}_3$ take their values as far as e' when multiplied by u_0, s_0 ;

$\bar{\mathfrak{a}}_3$, $\mathfrak{c}_3$, $\bar{\mathfrak{c}}_3$ are unity when multiplied by u_μ, s_μ.

The solution is of the form

$$u_\lambda\zeta^{-1} = \mathrm{a}\lambda\Sigma_i(\lambda_i\zeta^{2i+\tau}+\lambda'_i\zeta^{2i-\tau}), \text{ when } \tau \neq 0, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and

$$u_\lambda\zeta^{-1} = \mathrm{a}\lambda\Sigma_i\lambda_i\zeta^{2i}, \qquad \text{when } \tau = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4')$$

The process consists in first finding the series for A in each case, and then, after the substitution of the solution in the differential equation, to form the equations of condition for the unknown coefficients. These are solved by continued approximation, there being one pair of equations of principal importance in finding each pair of coefficients λ_i, λ'_{-i}. The known values c_0, g_0 of c, g, and the definitions of e, k given in Chap. III. are sufficient. The further definition of the linear constant (§ 24) occurs in sections (ii), (iv), (v), (viii) below.

53. The general type of the equation for z, for terms with characteristic λ and arguments $2i+\tau$, is, by Chap. I., § 20,

$$D^2 z_\lambda - 2Mz_\lambda = \mathrm{a}\lambda A\iota \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where

$$A = \Sigma_i A_i(\zeta^{2i+\tau} - \zeta^{-2i-\tau}).$$

Here, by Chap. I., equation (18′),

$$\mathrm{a}\lambda A\iota = \text{Part, char}^{\mathrm{c}}.\ \lambda, \text{ in}\left[-\frac{1}{2}\frac{\partial \Omega_1}{\partial z}\right.$$

$$\left. -\frac{3}{2}\frac{\Sigma z_\mu}{\mathrm{a}}(P\Sigma u_\mu \zeta^{-1} + P\Sigma s_\mu \zeta)\right] \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\Sigma u_\mu = u_e + u_{e'} + u_a, \qquad \Sigma z_\mu = z_k.$$

The first term of equation (18′) contributes nothing to the terms of the second order.

Also, by Chap. I., Sect. (iii), substituting for u, s, z the values just given and neglecting powers and products of u_μ, s_μ, z_μ,

$$-\frac{1}{2}\frac{\partial \Omega_1}{\partial z} = \mathrm{m}^2 z_k \mathfrak{h}_2 + \frac{3}{2}\frac{\mathrm{m}^2}{a'} z_k(u_0 + s_0) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $\mathfrak{h}_2$ takes its value as far as e'.

The solution is

$$z_\lambda \iota = \mathrm{a}\lambda \Sigma_i \lambda_i(\zeta^{2i+\tau} - \zeta^{-2i-\tau}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The process is the same as before. There is only one equation of principal importance in finding any coefficient λ_i. The known values c_0, g_0 of c, g and the definitions of the arbitrary constants given in Chap. III. are sufficient.

54. The following table gives the various classes of terms of the second order, with the sections in which they are considered below.

Section.	λ.	Arguments.	Type of Coefficients.
(ii)	e^2	$2i\pm 2c$, $2i$	(ϵ^2), (ϵ'^2), $(\epsilon\epsilon')$
(iii)	ee'	$2i\pm(c+m)$, $2i\pm(c-m)$	$(\epsilon\eta)$, $(\epsilon'\eta')$, $(\epsilon\eta')$, $(\epsilon'\eta)$
(iv)	e'^2	$2i\pm 2m$, $2i$	(η^2), (η'^2), $(\eta\eta')$
(v)	k^2	$2i\pm 2g$, $2i$	(k^2), (k'^2), (kk')
(vi)	ea	$2i_1\pm c$	(ϵa), $(\epsilon' a)$
(vii)	$e'a$	$2i_1\pm m$	(ηa), $(\eta' a)$
(viii)	a^2	$2i$	(a^2)
(ix)	ke	$\pm(2i+g+c)$, $\pm(2i+g-c)$	$\pm(k\epsilon)$, $\pm(k\epsilon')$
(x)	ke'	$\pm(2i+g+m)$, $\pm(2i+g-m)$	$\pm(k\eta)$, $\pm(k\eta')$
(xi)	ka	$\pm(2i_1+g)$	$\pm(ka)$

where

$$2i=0,\quad \pm 2,\quad \pm 4\ldots,\qquad 2i_1=\pm 1,\quad \pm 3,\quad \pm 5\ldots$$

The coefficients in the last column have the suffixes i or i_1, when the corresponding arguments have them, that is, according as the coefficients do not or do contain the first power of a. It will be remembered that when the characteristic contains an odd power of k (that is, for the terms in z), $(k'\epsilon')_i=-(k\epsilon)_{-i}$, $(k'\epsilon)_i=-(k\epsilon')_{-i}$, etc. Hence, Sections (ii)–(viii) contain all second order terms in u, Sections (ix)–(xi) all second order terms in z.

The results selected for publication are the numerical values of A_i, A'_i and those of the unknowns λ_i, λ'_i.

The degree of accuracy to which the various results have been carried depend, first, on the general numerical magnitude of the characteristic; secondly, for A_i, A'_i, on the cases where $2i\pm\tau$, $2i\pm\tau\pm c$, or $2i\pm\tau\pm g$ become small; thirdly, for λ_i, λ'_i, on the accuracy required for certain terms of higher orders. The approximate numerical magnitudes of the characteristics are given by

$$e=\cdot 11,\qquad e'=\cdot 017,\qquad k=\cdot 045,\qquad a=\cdot 0026.$$

For further remarks on the numerical results, see Chap. I., Sect. (viii).

References to previously published results are given in all cases.

Section (ii). *Characteristic* e^2.

55. Here $\lambda = e^2$, and $2i+\tau$ has the values $2i \pm 2c$ forming one set of equations of condition and the value $2i$ forming another set. The values of A corresponding to the two sets are obtained from equation (2) of this chapter, and they are given in the following table :—

Values of A. Coefficients of

i.	ζ^{2i+2c}.	ζ^{2i-2c}.	ζ^{2i}.
5	+ ·00000 0002	+ ·00000 0397	+ ·00000 0059
4	+ ·00000 0146	+ ·00002 4815	+ ·00000 4028
3	+ ·00001 0222	+ ·00125 4227	+ ·00025 3358
2	+ ·00064 6617	+ ·03959 7603	+ ·01291 0151
1	+ ·03321 1631	− ·08783 1650	+ ·41545 8343
0	+ 1·08874 2537	+ ·21926 5334	− ·46585 5712
−1	− ·10743 6306	+ ·00383 3333	+ ·08798 3630
−2	+ ·00888 0539	+ ·00005 3547	+ ·00151 7439
−3	+ ·00015 0316	+ ·00000 0664	+ ·00002 1196
−4	+ ·00000 2104	+ ·00000 0008	+ ·00000 0263
−5	+ ·00000 0026		+ ·00000 0003
Sum	+ 1·02420 7834	+ ·17619 8278	+ ·05229 2758

The solution is expressed by (Chap. I., Sect. (iv)),

$$u_{e^2}\zeta^{-1} = ae^2\Sigma_i\left[(\epsilon^2)_i\zeta^{2i+2c} + (\epsilon'^2)_i\zeta^{2i-2c} + (\epsilon\epsilon')_i\zeta^{2i}\right].$$

Solving the two sets of equations of condition—namely, that giving $(\epsilon^2)_i$, $(\epsilon'^2)_i$ and that giving $(\epsilon\epsilon')_i$—we obtain the values of these coefficients.

Values of

i.	$(\epsilon^2)_i$.	$(\epsilon'^2)_i$.	$(\epsilon\epsilon')_i$.
5	+·00000 00000 1	+·00000 00049	+·00000 00004 0
4	+·00000 00011 0	+·00000 04893	+·00000 00459 5
3	+·00000 01135 4	+·00004 84244	+·00000 47226 4
2	+·00001 16070 9	+·00428 5788	+·00046 03442
1	+·00112 37013	+·01564 7028	+·03917 99373
0	+·09402 3537	+·03180 1697	−·13311 2689
−1	−·06517 3271	+·00006 45654	+·01492 2756
−2	+·00133 0056	+·00000 06650	+·00002 21364
−3	+·00000 17404	+·00000 00057 4	+·00000 02603
−4	+·00000 00260	+·00000 00000 6	+·00000 00022 8
−5	+·00000 00003		+·00000 00000 2
Sum	+·03131 7512	+·05184 8668	−·07852 2483

These coefficients were given to eight places of decimals on pp. 325, 323 of my paper referred to in § 48, having been obtained by the use of the homogeneous equations. The notation is different. The symbols (ϵ^2), (ϵ'^2) are there denoted by $f/Y_0^2, f'/Y_0^2$ respectively; the symbol $(\epsilon\epsilon')_i$ used here is not the exact equivalent of $\delta a_i/Y_0^2$ in that paper, owing to the meaning there assigned to a_0 being different from that of a. To compare them we must put

$$(\epsilon\epsilon')_i=(a_0\ \delta a_i+a_i\ \delta a_0)\div Y_0^2\ a_0^2,$$

the terms in the right-hand member being the quantities contained in the paper referred to.

An error of one unit in the sixth place of the value of $(\epsilon^2)_{-2}$, or f_{-2} in the paper, was discovered, inducing smaller errors in the other coefficients. All the coefficients have been re-calculated by the method of this memoir and the results, as seen above, extended to nine places of decimals.

The long-period inequality with a small divisor is that having the coefficients $(\epsilon^2)_{-1}$, $(\epsilon'^2)_1$. This was separately calculated by the homogeneous equations as in the "Elliptic Inequalities" and the results were verified by the equations of condition which the above method furnishes.

Section (iii). *Characteristic* ee'.

56. Here $2i+\tau$ has the two sets of values $2i\pm(\mathrm{c}+\mathrm{m})$ and $2i\pm(\mathrm{c}-\mathrm{m})$. The corresponding values of A are given in the following tables.

Value of A. Coefficients of

i.	$\zeta^{2i+\mathrm{c}+\mathrm{m}}$.	$\zeta^{2i-\mathrm{c}-\mathrm{m}}$.
5	−·00000 0002	+·00000 0204
4	−·00000 0159	+·00001 2849
3	−·00001 0382	+·00066 4341
2	−·00056 4777	+·02214 1177
1	−·02154 4824	+·02587 20248
0	−·26294 86805	−·08121 00493
−1	+·04040 23983	−·00205 8170
−2	+·00196 1854	−·00005 0845
−3	+·00006 3492	−·00000 0753
−4	+·00000 0984	−·00000 0009
−5	+·00000 0013	
Sum ...	−·24264 0083	−·03462 9230

Value of A. Coefficients of

i.	$\zeta^{2i+\mathrm{c}-\mathrm{m}}$.	$\zeta^{2i-\mathrm{c}+\mathrm{m}}$.
5	+·00000 0016	−·00000 0032
4	+·00000 1031	−·00000 2037
3	+·00006 5570	−·00011 0038
2	+·00341 6192	−·00414 1602
1	+·11582 9253	−·04792 22751
0	+·23207 94821	+·09218 67678
−1	−·02089 28787	+·01113 8493
−2	−·00037 5417	+·00032 6072
−3	−·00000 9929	+·00000 4995
−4	−·00000 0149	+·00000 0066
−5	−·00000 0002	+·00000 0001
Sum ...	+·33011 3168	+·05148 0411

The solution is

$$u_{ee'}\zeta^{-1}=\mathrm{a}ee'\Sigma_i\left[(\epsilon\eta)_i\zeta^{2i+\mathrm{c}+\mathrm{m}}+(\epsilon'\eta')_i\zeta^{2i-\mathrm{c}-\mathrm{m}}+(\epsilon\eta')_i\zeta^{2i+\mathrm{c}-\mathrm{m}}+(\epsilon'\eta)_i\zeta^{2i-\mathrm{c}+\mathrm{m}}\right],$$

the first two terms forming one set of equations of condition, and the other two another set.

Values of

i.	$(\epsilon\eta)_i$.	$(\epsilon'\eta')_i$.
5		+·00000 00023
4	−·00000 00016	+·00000 02402
3	−·00000 01635	+·00002 40337
2	−·00001 60440	+·00223 19829
1	−·00143 5419	+·16122 2282
0	−·09352 2778	+·14515 115
−1	−·37910 7012	+·00011 3163
−2	−·00035 1502	−·00000 09258
−3	+·00000 14383	−·00000 00095
−4	+·00000 00149	−·c0000 00001
−5	+·00000 00001	
Sum ...	−·47443 1467	+·30874 192

i.	$(\epsilon\eta')_i$.	$(\epsilon'\eta)_i$.
5	+·00000 00001	−·00000 00003
4	+·00000 00101	−·00000 00346
3	+·00000 10020	−·00000 34589
2	+·00009 30969	−·00032 25311
1	+·00721 48506	−·02437 4803
0	+·12769 0229	−·22224 552
−1	+·03961 720	−·00035 7897
−2	+·00004 6236	+·00000 62531
−3	−·00000 01813	+·00000 00632
−4	−·00000 00020	+·00000 00006
−5		
Sum ...	+·17466 244	−·24729 793

The short period inequalities having small divisors are those with coefficients $(\epsilon\eta)_0$, $(\epsilon'\eta')_0$; $(\epsilon\eta)_{-1}$, $(\epsilon'\eta')_1$; $(\epsilon\eta')_0$, $(\epsilon'\eta)_0$; $(\epsilon\eta')_{-1}$, $(\epsilon'\eta)_1$. For the purpose of obtaining these with the required accuracy, the corresponding coefficients in A are carried one place further than the rest. The values of A have been computed by both (17), (17′) of § 20. The slow approximations to the values of the coefficients with suffix zero were avoided by the method of § 29.*

The values of the coefficients of the corresponding terms in the true longitude have been published in a note in the *Monthly Notices*, Vol. LV. p. 4.

Section (iv). *Characteristic e'^2.*

57. Here $2i+\tau$ has the two sets of values $2i\pm 2\text{m}$ and $2i$. The terms are similar in form to those of Section (ii).

Values of A. Coefficients of

i.	$\zeta^{2i+2\text{m}}$.	$\zeta^{2i-2\text{m}}$.	ζ^{2i}.
5		+·00000 003	−·00000 001
4	+·00000 003	+·00000 166	−·00000 052
3	+·00000 236	+·00008 791	−·00002 893
2	+·00010 062	+·00308 035	−·00113 096
1	+·00208 790	+·01194 981	−·01521 048
0	+·01086 3018	+·00684 6387	−·03584 293
−1	−·08426 889	−·00006 164	+·02561 810
−2	+·00013 885	+·00000 798	−·00005 617
−3	+·00000 733	+·00000 019	−·00000 228
−4	+·00000 013		−·00000 003
−5			
Sum ...	−·07106 865	+·02191 268	−·02665 421

* The results for the coefficients in A are not theoretically accurate in the last places of decimals given. The small divisors occurring in the coefficients mentioned and the other divisors are of such a size that the results for the coefficients $(\epsilon\eta)_0$, &c., are, however, theoretically accurate to the last place given in each case.

The solution is

$$u_{e'^2}\zeta^{-1}=ae'^2\Sigma_i[(\eta^2)_i\zeta^{2i+2m}+(\eta'^2)_i\zeta^{2i-2m}+(\eta\eta')\zeta\ ],$$

which gives two sets of equations of condition as in Section (ii).

Values of

i.	$(\eta^2)_i$.	$(\eta'^2)_i$.	$(\eta\eta')_i$.
5		+·00000 00003	−·00000 00001
4	+·00000 00004	+·00000 00290	−·00000 00083
3	+·00000 0035	+·00000 2938	−·00000 08202
2	+·00000 2205	+·00027 6586	−·00007 6201
1	−·00003 277	+·02192 232	−·00585 014
0	−·05446 177	+·07221 455	−·01024 957
−1	−·10598 405	+·00007 267	+·02515 958
−2	+·00001 0669	+·00000 0016	−·00000 1878
−3	+·00000 0238	+·00000 0003	−·00000 0060
−4	+·00000 00026		−·00000 00005
−5			
Sum ...	−·16046 544	+·09448 911	+·00898 090

The long-period inequality with a small divisor is that having the coefficients $(\eta^2)_0$, $(\eta'^2)_0$. To obtain it with sufficient accuracy, the homogeneous equation (7) or (42) of Chap. I. was calculated for $2i+\tau=2m$, and combined with one of the equations of condition of principal importance in finding these coefficients by the above method; the corresponding terms in A are carried one place further. The slowness of the approximations was avoided as before.

Section (v). *Characteristic* k^2.

58. Here $2i+\tau$ has the two sets of values, $2i\pm 2g$ and $2i$. The forms are similar to those of Sections (ii), (iv).

Values of A. Coefficients of

i.	ζ^{2i+2g}.	ζ^{2i-2g}.	ζ^{2i}.
5	+ ·00000 0001	+ ·00000 0005	+ ·00000 0005
4	+ ·00000 0098	+ ·00000 0373	+ ·00000 0420
3	+ ·00000 8104	+ ·00001 9128	+ ·00003 0535
2	+ ·00061 3392	+ ·00019 5705	+ ·00179 3676
1	+ ·03948 0728	− ·09576 3403	+ ·05619 8303
0	+ 1·75467 4677	+ 1·75605 5745	− 3·51593 8315
− 1	− ·11368 2769	+ ·02156 4305	+ ·09211 3861
− 2	+ ·00119 0891	+ ·00022 7392	+ ·00118 5620
− 3	+ ·00001 6227	+ ·00000 2257	+ ·00001 2707
− 4	+ ·00000 0178	+ ·00000 0022	+ ·00000 0126
− 5	+ ·00000 0002		+ ·00000 0001
Sum ...	+ 1·68230 1529	+ 1·68230 1529	− 3·36460 3061

The solution is

$$u_{k^2}\zeta^{-1} = \mathrm{a}k^2\Sigma_i[(k^2)_i\zeta^{2i+2g} + (k'^2)_i\zeta^{2i-2g} + (kk')_i\zeta^{2i}].$$

Values of

i.	$(k^2)_i$.	$(k'^2)_i$.	$(kk')_i$.
5		+ ·00000 00000 8	− ·00000 00000 1
4	+ ·00000 00000 2	+ ·00000 00096	− ·00000 00014 9
3	+ ·00000 00009 0	+ ·00000 13301	− ·00000 01983
2	+ ·00000 01113	+ ·00020 4729	− ·00002 78210
1	+ ·00001 40450	+ ·04329 3868	− ·00434 42967
0	+ ·00165 67611	+ ·98752 5842	− 1·00079 9130
− 1	− ·09302 7702	+ ·00150 88256	+ ·08149 6924
− 2	+ ·00081 6246	+ ·00000 58653	+ ·00009 25048
− 3	+ ·00000 14413	+ ·00000 00300	+ ·00000 03390
− 4	+ ·00000 00059	+ ·00000 00001 8	+ ·00000 00016 6
− 5	+ ·00000 00000 4		+ ·00000 00000 1
Sum ...	− ·09053 9090	+ 1·03254 0500	− ·92358 1678

These coefficients were given to seven places of decimals by Mr. P. H. Cowell on pp. 119, 117 of his paper referred to in § 31 above, being obtained by means of the homogeneous equations. With his values as a first approximation, I recalculated and extended them to nine places by the above method. Small errors in $(k^2)_{-1}$, $(k'^2)_1$ were found. These two coefficients are those of a long-period term with a small divisor. The former was obtained by means of equation (44) of Chap. I., the latter being then found from one of the two equations of condition furnished by the general method used for the rest of the coefficients. The coefficients $(kk')_i$ in Mr. Cowell's paper are such that $(kk')_0=0$; to compare them with those given here, a transformation like that noted at the end of Sect. (ii) of this chapter must be made. The value of δa_0 in that formula is the $(kk')_0$ of the table in this section.

Section (vi). *Characteristic* eα.

59. Here $2i+\tau=2i_1\pm c$ where $2i_1$ is an odd positive or negative integer. It is not necessary to insert the suffix of i_1 in the tables.

Value of A. Coefficients of

$2i$.	ζ^{2i+c}.	ζ^{2i-c}.
9	+·00000 002	+·00000 010
7	+·00000 053	−·00000 676
5	−·00003 223	−·00135 769
3	−·00682 665	−·10010 7438
1	−·51730 75492	+·13499 37257
−1	+·14791 32045	−·10012 41023
−3	−·00226 4428	−·00655 106
−5	−·00141 822	−·00004 061
−7	−·00000 864	−·00000 022
−9	−·00000 004	
Sum ...	−·37994 400	−·07319 405

The solution is

$$u_{e\alpha}\zeta^{-1}=ae\alpha\Sigma_i[(\epsilon\alpha)_i\zeta^{2i+c}+(\epsilon'\alpha)_i\zeta^{2i-c}], \qquad 2i \text{ odd.}$$

Values of

$2i$.	$(\epsilon\alpha)_i$.	$(\epsilon'\alpha)_i$.
9	+·00000 00001	+·00000 00007
7	+·00000 00121	−·00000 0210
5	+·00000 0067	−·00006 8451
3	−·00018 3543	−·01423 8397
1	−·04675 5060	−·13023 797
−1	+·19695 832	−·01226 5219
−3	+·01753 2892	−·00059 1716
−5	−·00014 5521	−·00000 1318
−7	−·00000 0318	−·00000 00010
−9	−·00000 00008	
Sum ...	+·16740 685	−·15740 328

The long-period inequality is that having $(\epsilon\alpha)_{-\frac{1}{2}}$, $(\epsilon'\alpha)_{\frac{1}{2}}$ as coefficients. The homogeneous equation (42) of Chap. I. with $2i+\tau=-1+c$ was used with one of the two ordinary equations of condition giving these two coefficients, the other being, as usual, used as a control. The slow progress of the approximations was avoided as before.

Section (vii). *Characteristic* $e'\alpha$.

60. Here $2i+\tau=2i_1\pm m$. The suffix of i_1 will be omitted.

Value of A. Coefficients of

$2i$.	ζ^{2i+m}.	ζ^{2i-m}.
9		+·00000 003
7	+·00000 012	−·00000 138
5	+·00004 540	−·00033 248
3	+·00405 8824	−·02654 623
1	+·06010 42164	−·05865 88010
−1	−·04094 6441	+·01713 45698
−3	−·06584 676	+·01489 4425
−5	−·00036 752	+·00006 690
−7	−·00000 211	+·00000 036
−9		
Sum ...	−·04295 427	−·05344 261

The solution is

$$u_{e'}\,\zeta^{-1}=\mathrm{a}e'\alpha\Sigma_i[(\eta\alpha)_i\zeta^{2i+\mathrm{m}}+(\eta'\alpha)_i\zeta^{2i-\mathrm{m}}],\qquad 2i\text{ odd.}$$

Values of

$2i$.	$(\eta\alpha)_i$.	$(\eta'\alpha)_i$.
9	+·00000 00010	+ ·00000 00016
7	+·00000 0123	+ ·00000 0126
5	+·00001 8262	+ ·00000 8022
3	+·00286 0184	+ ·00008 9662
1	+·51611 841	− ·02661 542
−1	+·03082 496	−1·51100 393
−3	−·01661 267	+ ·00301 4686
−5	−·00001 8246	+ ·00000 4431
−7	−·00000 0037	+ ·00000 0015
−9	+·00000 00001	+ ·00000 00001
Sum ...	+·53319 099	−1·53450 241

The short-period inequalities are those having the coefficients $(\eta\alpha)_{\frac{1}{2}}$, $(\eta'\alpha)_{-\frac{1}{2}}$; $(\eta\alpha)_{-\frac{1}{2}}$, $(\eta'\alpha)_{\frac{1}{2}}$; the former pair, owing to the near coincidence of $1+\mathrm{m}$ with c_0, having a very small divisor. The corresponding values of the coefficients in A are carried to more places and the slow progress of the approximations to the first pair was avoided as before.

Section (viii). *Characteristic* α^2.

61. Here $\tau=0$.

Value of A. Coefficients of

i.	ζ^{2i}.
4	−·00000 01
3	−·00001 02
2	−·00025 53
1	+·05977 63
0	−·03355 85
−1	+·00220 05
−2	−·01280 45
−3	−·00006 20
−4	−·00000 04
Sum ...	+·01528 58

The solution is

$$u_{a^2}\zeta^{-1} = \mathrm{a}a^2\Sigma_i(a^2)_i\zeta^{2i}.$$

Values of

i.	$(a^2)_i$.
4	$+\cdot 00000\ 0007$
3	$+\cdot 00000\ 097$
2	$+\cdot 00009\ 385$
1	$+\cdot 00722\ 77$
0	$-\cdot 00960\ 28$
−1	$-\cdot 00720\ 42$
−2	$-\cdot 00142\ 707$
−3	$-\cdot 00000\ 223$
−4	$-\cdot 00000\ 0008$
Sum ...	$-\cdot 01091\ 38$

These coefficients were obtained to six places of decimals on p. 157 of my paper referred to in § 50 above by using the homogeneous equations. They have been recalculated by the method of this chapter and extended to seven places. In comparing the earlier results with those given here, the transformation noted at the end of Sect. (ii) of this chapter must be made. The errors in the earlier results were very small.

62. Having finished the terms of the second order in u, we now come to those in z which are distinguished by having the first power of k in all their characteristics. For these terms the equations (5)–(8) of Sect. (i) of this chapter are used. The calculations cause very little trouble and are not long. In the short-period inequalities with small divisors, the progress of the approximations is not very slow. Long-period inequalities do not produce small divisors.

Section (ix). *Characteristic* ke.

63. Here $2i+\tau$ takes the two sets of values $\pm(2i+\mathrm{g}+\mathrm{c})$ and $\pm(2i+\mathrm{g}-\mathrm{c})$, each set giving an independent set of equations of condition for the coefficients.

Values of A. Coefficients of

i.	ζ^{2i+g-c}.	ζ^{2i+g-c}.
5	+ ·00000 0001	+ ·00000 0010
4	+ ·00000 0054	+ ·00000 0817
3	+ ·00000 4225	+ ·00006 2854
2	+ ·00032 0242	+ ·00398 8176
1	+ ·02045 4910	+ ·16874 9622
0	+ ·87305 5636	+ ·86734 1740
−1	+ ·13526 5495	− ·01316 9637
−2	− ·00250 6140	− ·00046 2414
−3	− ·00008 9740	− ·00000 7977
−4	− ·00000 1579	− ·00000 0106
−5	− ·00000 0019	− ·00000 0001
Sum ...	+1·02650 3085	+1·02650 3084

Coef. of $\zeta^{2i-g-c} = -$ coef. of $\zeta^{-2i+g-c}$,

Coef. of $\zeta^{2i-g+c} = -$coef. of $\zeta^{-2i+g-c}$.

The solution is

$$z_{ke}\iota = ake\Sigma_i[(\epsilon k)_i\zeta^{2i+g+c} + (\epsilon' k')_i\zeta^{2i-g-c} + (\epsilon' k)_i\zeta^{2i+g-c} + (\epsilon k')_i\zeta^{2i-g+c}]$$
$$= ake\Sigma_i[(\epsilon k)_i(\zeta^{2i+g+c} - \zeta^{-2i-g-c}) + (\epsilon' k)_i(\zeta^{2i-g-c} - \zeta^{-2i-g+c})].$$

Values of

i.	$(\epsilon k)_i$.	$(\epsilon' k)_i$.
5		+·00000 00001 1
4	+·00000 00006 0	+·00000 00149 3
3	+·00000 00740 3	+·00000 20814 9
2	+·00001 00784 2	+·00030 77463
1	+·00146 61399	+·05543 3972
0	+·25091 3591	−·73687 7762
−1	−·11999 3099	−·00811 4895
−2	−·00179 7078	−·00004 46815
−3	−·00000 93584	−·00000 03014 9
−4	−·00000 00626 5	−·00000 00021 4
−5	−·00000 00004 1	−·00000 00000 1
Sum ...	+·13059 0286	−·68929 3827

There is no short-period inequality with a small divisor.

Section (x). *Characteristic* ke′.

64. Here $2i+\tau$ has the two sets of values $\pm(2i+g+m)$ and $\pm(2i+g-m)$, giving rise to two independent sets of equations of condition.

Values of A. Coefficients of

i.	ζ^{2i+g+m}.	ζ^{2i+g-m}.
5		+·00000 0002
4	−·00000 0034	+·00000 0222
3	−·00000 2530	+·00001 7263
2	−·00016 8839	+·00114 5257
1	−·00794 5172	+·05318 2849
0	−·00150 67489	−·00386 17596
−1	+·05329 66967	−·00788 15823
−2	−·00090 3017	+·00013 7147
−3	−·00002 6532	+·00000 3923
−4	−·00000 0433	+·00000 0061
−5	−·00000 0006	+·00000 0001
Sum ...	+·04274 3385	+·04274 3383

Coef. of $\zeta^{2i-g-m} = -$coef. of $\zeta^{-2i+g+m}$,
Coef. of $\zeta^{2i-g+m} = -$coef. of $\zeta^{-2i+g-m}$.

The solution is

$$z_{ke'\iota} = ake'\Sigma_i[(\eta k)_i(\zeta^{2i+g+m} - \zeta^{-2i-g-m}) + (\eta' k)_i(\zeta^{2i+g-m} - \zeta^{-2i-g+m})].$$

Values of

i.	$(\eta k)_i$.	$(\eta' k)_i$.
5		+·00000 00000 2
4	−·00000 00004 5	+·00000 00030 2
3	−·00000 00549 5	+ 00000 03925 4
2	−·00000 71590	+·00005 16918
1	−·00092 26112	+·00680 83683
0	−·01600 9252	+·01924 4655
−1	−·11002 6721	+·04091 0818
−2	−·00033 46147	+·00008 41128
−3	−·00000 20106	+·00000 04271
−4	−·00000 00139 3	+·00000 00026 7
−5	−·00000 00001 0	+·00000 00000 1
Sum ...	−·12730 2438	+·06710 0471

The short-period inequalities having small divisors are those with coefficients $(\eta k)_0$, $(\eta k)_{-1}$, $(\eta' k)_0$, $(\eta' k)_{-1}$; for these the corresponding values of A_i are carried to ten places of decimals.

Section (xi). *Characteristic* kα.

65. Here $2i+\tau = \pm(2i_1+\mathrm{g})$. The suffix of i_1 will be omitted.

Value of A. Coefficients of

$2i$.	$\zeta^{2i+\mathrm{g}}$.
9	+·00000 0008
7	+·00000 0534
5	+·00002 6082
3	−·00008 7162
1	−·19302 7070
−1	−·18559 9847
−3	+·00734 3267
−5	+·00002 8303
−7	−·00000 0473
−9	−·00000 0013
Sum ...	−·37131 6371

Coef. of $\zeta^{2i-\mathrm{g}} = -$coef. of $\zeta^{-2i+\mathrm{g}}$.

The solution is

$$z_{\mathrm{k}\alpha} = \mathrm{ak}\alpha\Sigma_i(k\alpha)_i(\zeta^{2i+\mathrm{g}} - \zeta^{-2i-\mathrm{g}}), \quad 2i \text{ odd.}$$

Values of

$2i.$	$(k\alpha)_i.$
9	$+\cdot00000 \quad 00000 \quad 7$
7	$+\cdot00000 \quad 00072 \quad 0$
5	$+\cdot00000 \quad 05027$
3	$-\cdot00005 \quad 33463$
1	$-\cdot06026 \quad 0507$
-1	$+\cdot15913 \quad 4186$
-3	$+\cdot00375 \quad 4653$
-5	$+\cdot00000 \quad 67585$
-7	$+\cdot00000 \quad 00082$
-9	$-\cdot00000 \quad 00000 \quad 9$
Sum ...	$+\cdot10258 \quad 2262$

There are no short-period inequalities with small divisors.

(*To be continued.*)

Appendix.—The terms of the third order (Chap. V.) are in process of calculation. The following results, properly belonging to Chap. V., have already been obtained and may be recorded here. They have been used as the basis of a paper in the *Monthly Notices*, "On the Mean Motions of the Lunar Perigee and Node" (1897 March).

Let

$$c = c_0 + e^2 c_{e^2} + e'^2 c_{e'^2} + k^2 c_{k^2},$$
$$g = g_0 + e^2 g_{e^2} + e'^2 g_{e'^2} + k^2 g_{k^2}.$$

Then

$$c_{e^2} = +\cdot00268 \quad 575, \qquad g_{e^2} = +\cdot00318 \quad 579,$$
$$c_{e'^2} = -\cdot03465 \quad 53, \qquad g_{e'^2} = +\cdot00564 \quad 65,$$
$$c_{k^2} = +\cdot05384 \quad 91, \qquad g_{k^2} = -\cdot00806 \quad 633.$$

The value of g_{k^2} is quoted from Mr. Cowell's paper referred to in § 31 above.

Haverford College, Pa. U.S.A.:
1897 *May* 12.

Theory of the Motion of the Moon; containing a New Calculation of the Expressions for the Coordinates of the Moon in Terms of the Time. By Ernest W. Brown, M.A., Sc.D., F.R.S.

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PART II. CHAPTER V.

In the following pages I continue the Memoir the first part of which was published under the same title in the *Memoirs of the Royal Astronomical Society* in 1897. The general theory was given in Chap. I. as completely as I could then foresee would be necessary for the whole work. In Chaps. II., III., IV. the numerical results up to and inclusive of the terms of the second order were given.

As the work progressed modifications tending to simplify or abbreviate the calculations naturally occurred. These, however, were fewer than might have been expected. The most important of them is given in Section (ii) below, consisting of a new method for finding the values of the final coefficients, after those of the quantities denoted in Chap. I. by A have been obtained. Previously this process consisted in solving, by continued approximation, for each characteristic and argument, a set of linear equations which were generally about 20 in number, with 20 unknowns. This process I had not succeeded in arranging conveniently for the computer, and as, in the terms of the third order, it involved about one-third of the whole work, some change was desirable. The investigation which led up to the new

method for this purpose was made from a different point of view some three years ago; its usefulness became apparent directly the arrangement of the work for the computer was under consideration. Moreover, the numerical errors made in solving the linear equations were quite numerous; under the method of Section (ii) they have been no more frequent than in other parts of the work.

The numerical results given below are the values for all the terms of the third order, with certain subsidiary results which the above-mentioned modifications require. The degree of accuracy which the theory up to this point attains may be best appreciated by a statement of the maximum number of coefficients of the fourth and higher orders, which may be as great as 1″ of arc in longitude. Of the fourth order, one of 4″ and three of 2″, which contain only e, k in their characteristics, and twenty of 1″; of the fifth order there are two of 1″, which involve both e, k. Moreover, the principal parts of nearly all of these are the purely elliptic terms.

I have received very great help in performing the calculations from Mr. Ira I. Sterner, A.B., of Haverford College, who has, since 1897 September, been my only assistant. That so much has been achieved in the time we were able to give is largely due to his accuracy and capability. Much of the work done by him would scarcely have been attempted by an ordinary computer without very extended instructions, while his knowledge of arithmetical processes has not only been a great saving of time and labour, but has made the chief part of my task—that of testing and correcting his work—a comparatively light one.*

The following is the table of contents of Chap. V.:—

Section (i). A brief outline of the application of the general method to the terms of the third order in the calculation of the series *A*.

Section (ii). New method for solving the linear equations when the series *A* have been obtained. Numerical values of certain quantities required in this method.

Section (iii). Modification of this method, in order to avoid, as far as possible, the loss of accuracy arising with long-period terms.

Section (iv). The method of calculating the new parts of the motions of

* A portion of the expense of making the computations necessary to obtain the results given below has been met by a grant from the Government Grant Fund of the Royal Society.

the perigee and node, and the coefficients arising therewith. Numerical values of certain quantities required.

Section (v). The final numerical results for the series A, and for the coefficients of all terms of the third order in u, z.*

Section (i). *Formulæ and Tests.*

66. About two-thirds of the whole labour of obtaining the coefficients of the third order consists in the calculation of A. The products in the third line of equation (17) and of the second line of (18) in Chap. I., are formed by putting

$$u_\mu = u_1 + u_2,\ s_\mu = s_1 + s_2,\ z_\mu = z_1 + z_2$$

and choosing out the parts of the third order. The products in the fourth and fifth lines of (17) and the third line of (18) are obtained with

$$u_\mu = u_1,\ s_\mu = s_1,\ z_\mu = z_1$$

Here

$$u_1 = u_{\mathrm{e}} + u_{e'} + u_a,\ z_1 = z_{\mathrm{k}},$$
$$u_2 = u_{\mathrm{e}^2} + u_{\mathrm{e}e'} + u_{e'^2} + u_{\mathrm{k}^2} + u_{\mathrm{e}a} + u_{e'a} + u_{a^2},$$
$$z_2 = z_{\mathrm{ke}} + z_{\mathrm{k}e'} + z_{\mathrm{k}a},$$

the expression for s_2 being similar to that for u_2.

The parts arising from Ω_1, namely, $\frac{\partial \Omega_1}{\partial s}\zeta^{-1}$, $-\frac{1}{2}\frac{\partial \Omega_1}{\partial z}$, are treated in like manner.

The parts arising from the first terms in the right-hand members of the equations, are treated in Section (iv) below; they only appear when the terms of arguments $2i \pm \mathrm{c}$, $2i \pm \mathrm{g}$ are under consideration.

The general method of procedure has been as follows:—The computer having performed the calculations allotted to him, I go over them all and test them by all the means available. Each multiplication of series is tested by sums as explained in Section (viii), Chap. I. The final values of A are tested in the same manner, but in larger groups, so as to make certain that no series of terms has been omitted; for this, the values of A for each character-

* It is intended to give the subsidiary results in a final chapter, or chapters, when the whole theory has been completed.

istic, with $\zeta=1$, are all added together and the sums compared with the corresponding sums obtained as directly as possible from the algebraical formulæ, which become quite simple when $\zeta=1$. It is true that this method will not test for all kinds of errors, *e.g.* the accidental interchange of u and s, but reliance has been placed less on test equations than on the care which has been taken to avoid errors. The calculations made after the method of Section (ii) were treated in the same way as far as possible; a further and very searching test was obtained by forming D^2u_λ, Du_λ with $\zeta=1$, for each value of τ and substituting the results in equation (24) of Chap. I. In the cases where calculations were not turned over to the computer, they were gone over again after an interval of time and, when possible, tested. Increased accuracy in making the calculations was always found whenever blocks of the same nature were performed together.

Section (ii). *Method of solving the Linear Differential Equations.*

67. The equation (24) of Chap. I. to be solved is

$$(D+\mathrm{m})^2 u_\lambda + M u_\lambda + N s_\lambda \zeta^2 = \mathrm{a}\lambda A \zeta = A' \quad . \quad . \quad . \quad . \quad (1)$$

Let

$$\Sigma\epsilon_i \zeta^{2i+1+\mathrm{c}_0} = \mathrm{u}_1', \qquad \Sigma\epsilon_i' \zeta^{2i+1-\mathrm{c}_0} = \mathrm{u}_2', \qquad D u_0 = \mathrm{a}\mathrm{u}_3',$$
$$\Sigma\epsilon'_{-i} \zeta^{2i-1+\mathrm{c}_0} = \mathrm{s}_1', \qquad \Sigma\epsilon_{-i} \zeta^{2i-1-\mathrm{c}_0} = \mathrm{s}_2', \qquad D s_0 = \mathrm{a}\mathrm{s}_3'.$$

Then u_j', s_j', $j=1, 2, 3$, form three particular integrals of (1) when $A=0$. In a paper to be published elsewhere, I have shown that the fourth particular integral can be given the forms

$$u = \mathrm{u}_4' = \mathrm{u}_3' D^{-1}\left(\frac{\mathrm{u}_1' D\mathrm{u}_2' - \mathrm{u}_2' D\mathrm{u}_1'}{\mathrm{u}_3'\mathrm{s}_3'}\right)$$
$$= \frac{1}{2}\mathrm{u}_3' \left[\frac{\mathrm{s}_1'\mathrm{u}_2' - \mathrm{u}_1'\mathrm{s}_2'}{\mathrm{u}_3'\mathrm{s}_3'} + D^{-1}\left\{\frac{\mathrm{C}_{12}}{\mathrm{u}_3'\mathrm{s}_3'} - \frac{\mathrm{s}_1'\mathrm{u}_2' - \mathrm{u}_1'\mathrm{s}_2'}{\mathrm{u}_3'\mathrm{s}_3'}\left(2\mathrm{m} + \frac{D\mathrm{u}_3'}{\mathrm{u}_3'} - \frac{D\mathrm{s}_3'}{\mathrm{s}_3'}\right)\right\}\right]$$
$$s = \mathrm{s}_4' = \overline{\mathrm{u}_4'},$$

where C_{12} is a constant given by

$$\mathrm{C}_{12} = \mathrm{s}_2' D\mathrm{u}_1' - \mathrm{u}_1' D\mathrm{s}_2' + \mathrm{u}_2' D\mathrm{s}_1' - \mathrm{s}_1' D\mathrm{u}_2' - 2\mathrm{m}(\mathrm{s}_1'\mathrm{u}_2' - \mathrm{u}_1'\mathrm{s}_2')$$
$$= 2\Sigma(2i+1+\mathrm{m}+\mathrm{c}_0)\,{}^{2}_{i} + 2\Sigma(2i-1-\mathrm{m}+\mathrm{c}_0)\epsilon_{-i}'^{2},$$

the bar over u_4' denoting that ζ^{-1} has been put for ζ, that is $-\iota$ for ι in the expression for u_4'.

It is further shown that

$$\frac{u_4'}{C_{12}} = u_3' D^{-1} q + u_4\zeta = u_3' \iota (n-n')(t-t_0) q + u_4\zeta,$$

where q is the constant term under the sign D^{-1} in the expansion of u_4'/C_{12} in powers of ζ^2 and $u_4\zeta$ is a series of the same form as u_3.

Finally, it is shown that the solution of (1) when A is not zero can be put into the form

$$u_\lambda = \frac{1}{C_{12}}\Big[u_1' D^{-1}(s_2' A' + u_2' \bar{A}') - u_2' D^{-1}(s_1' A' + u_1' \bar{A}')\Big] + u_4\zeta D^{-1}(s_3' A' + u_3' \bar{A}')$$
$$- u_3' D^{-1}\{(s_4\zeta^{-1} A' + u_4\zeta \bar{A}') - q D^{-1}(s_3' A' + u_3' A')\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

in which the bar over A has the same meaning as before.

This is the form required ; it will be noticed at once that most of th operations will consist of multiplications of series, and will therefore be i line with the work which was necessary for the calculation of A.

68. It is advisable to make a few changes in order that the formula may be more compact. We first observe that for all purposes except the parts due to ☊, we require, not u_λ, but u_λ/u_0; the parts due to ☊ have the factor m^2, and are therefore small, and the multiplication of u_λ/u_0 by u_0 is easier than that of u_λ by $1/u_0$ owing to the fact that u_0 has one coefficient unity. As there is no increase of trouble involved in finding u_λ/u_0, we shall do so.

Write therefore—

$$u_1 = \frac{u_1'\zeta^{-1-c_0}}{\epsilon_0'}, \quad u_2 = \frac{u_2'\zeta^{-1+c_0}}{\epsilon_0'}, \quad s_1 = \frac{s_1'\zeta^{1-c_0}}{\epsilon_0'}, \quad s_2 = \frac{s_2'\zeta^{1+c_0}}{\epsilon_0'} \ldots \quad \ldots \quad \ldots \quad (3)$$

$$u_3 = u_3'\zeta^{-1}, \quad s_3 = s_3'\zeta \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$U_1 = \frac{a\epsilon_0' u_1'\zeta^{-c_0}}{C_{12}u_0}, \quad U_2 = \frac{a\epsilon_0' u_2'\zeta^{c_0}}{C_{12}u_0}, \quad U_3 = \frac{a u_3'}{u_0}, \quad U_4 = \frac{a u_4\zeta}{u_0} \ldots \quad \ldots \quad \ldots \quad (5)$$

These ten quantities are then all series of the form $\Sigma\, p_i\, \zeta^{2i}$ with numerical coefficients, and the constant coefficients in u_2, s_3, U_3, are all unity.

Also, $D^{-1}(X\zeta^c)$ may be written $\zeta^c (D+c)^{-1}X$; the operation $D^{-1}\zeta^\tau$ consisting of a division by τ, the operation $(D+c)^{-1}\zeta^\tau$ will consist of a division by $\tau + c$.

The formula (2) may now be written

$$\frac{1}{\lambda}\frac{u_\lambda}{u_0} = U_1(D-c_0)^{-1}(s_2 A + u_2\bar{A}) - U_2(D+c_0)^{-1}(s_1 A + u_1\bar{A}) + U_4 D^{-1}(s_3 A + u_3\bar{A})$$
$$- U_3 D^{-1}\{(s_4 A + u_4\bar{A}) - q D^{-1}(s_3 A + u_3\bar{A})\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Next, put

$$\left.\begin{array}{ll} Q_\lambda=(D-c_0)^{-1}(s_2A+u_2\bar{A}), & V_\lambda=D^{-1}(s_3A+u_3\bar{A}), \\ W_\lambda=s_4A+u_4\bar{A}, & T_\lambda=D^{-1}(qV_\lambda-W_\lambda) \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Then since

$$\bar{s}_1=u_2, \qquad \bar{u}_1=s_2, \qquad \bar{u}_3=-s_3, \qquad \bar{u}_4=s_4, \qquad \bar{D}=-D,$$

and therefore

$$-(D+c_0)^{-1}(s_1A+u_1\bar{A})=\bar{Q}_\lambda, \quad u_3\bar{A}=-\overline{s_3A}, \quad u_1\bar{A}=\overline{s_4A}, \quad \bar{V}_\lambda=V_\lambda, \quad \bar{W}_\lambda=W_\lambda, \quad \bar{T}_\lambda=-T_\lambda,$$

the equation (6) takes the final form

$$\frac{1}{\lambda}\frac{u_\lambda}{u_0}=U_1Q_\lambda+U_2\bar{Q}_\lambda+U_3T_\lambda+U_4V_\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

69. The calculations are arranged as follows :—The series u_2, s_2, s_3, s_4, U_1, U_2, U_3, U_4 are obtained once for all by multiplication or "special values," and the logarithms of the coefficients written out on slips, as explained in Chap. I., Section (viii). The slips containing A having been made out for each value of λ, the multiplications

$$s_2A,\ u_2\bar{A},\ s_3A,\ s_4A \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

are performed by the computer, and thence the values of Q_λ, V_λ, T_λ are, by a few easy processes, obtained. The slips containing these latter are then made out, those for Q_λ serving for $\bar{Q}_\lambda$. Finally the multiplications

$$U_1Q_\lambda,\ U_2\bar{Q}_\lambda,\ U_3T_\lambda,\ U_4V_\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

are performed by the computer, and thence, by addition, the value of u_λ/u_0 is obtained.

The series A are of the form $\Sigma_i p_i \zeta^{2i+\tau}+\Sigma_i p_i \zeta^{2i-\tau}$. When τ is not zero, the process for each such double set involves sixteen multiplications of series of the form $(\Sigma p_i\zeta^{2i})(\Sigma q_i\Sigma^{2i})$. When $\tau=0$, half the number suffices. When $\tau=c$, a new part of the motion of the perigee is under consideration, and though the same process may be used, it is not convenient (see Section (iv) below).

When $\tau=0$ and $2i$ is even, it would appear that T_λ gives rise to terms with the time as a factor. It is shown in the paper referred to at the beginning of this section that such terms can always be made to disappear. However, none such occur amongst the terms of the third order. In any

case the method of this section will not be used for them, as the approximations in the ordinary method are rapid, owing to the absence of any small divisors.

The same method may be applied, using well-known formulæ, to the calculation of z_λ, but it appeared easier, at any rate in the terms of the third order, to use the method given in Chap. I.

The chief objection to the method consists in the fact that the small coefficients which accompany large values of i appear as differences between comparatively large numbers. This fact does not impair the required accuracy of the results, but the multiplications are much longer than they otherwise would be. Thus the coefficient of ζ^{-6-3c} in $u_{e^3}/e^3 u_0$ appears as follows :—

From $U_1 Q$,	$-$·00006	418
,, $U_2 \bar{Q}$,	$+$·00006	496
,, $U_3 T$,	$+$·00000	316
,, $U_4 V$,	$-$·00000	392
Sum	$+$·00000	002

On the other hand, as the presence of this difficulty can be shown to be peculiar to the method, it furnishes a means of detecting a certain class of error.

70. The following are the numerical results required for the method of this Section :—

Values of the coefficients of ζ^{2i} in

i	s_2.	u_2.	s_3.	s_4.
4		$-$ ·00000 002		
3	$-$·00000 017	$-$ ·00000 279	$+$ ·00000 0012	$-$·00000 003
2	$+$·00006 960	$-$ ·00041 211	$+$ ·00000 0491	$-$·00000 147
1	$+$·19895 557	$-$ ·07424 980	$-$ ·00869 5747	$+$·03592 927
0	$-$·33619 118	$+$1	$-$1	$-$·93146 358
$-$1	$-$·00196 357	$+$ ·00016 930	$-$ ·00454 7122	$-$·00648 148
$-$2	$-$·00001 349	$-$ ·00000 090	$-$ ·00002 9393	$-$·00004 410
$-$3	$-$·00000 010	$-$ ·00000 001	$-$ ·00000 0210	$-$·00000 032
$-$4			$-$ ·00000 0002	
Sum	$-$·13914 334	$+$ ·92550 367	$-$1·01327 1971	$-$·90206 171

Values of the Coefficients of ζ^{2i} in

i	U_1.	U_2.	U_3.	U_4
4		− ·00000 003		
3	−·00000 015	− ·00000 417	+ ·00000 013	−·00000 022
2	−·00002 034	− ·00066 197	+ ·00001 892	−·00003 094
1	−·00317 658	− ·16544 214	+ ·00303 148	−·00506 996
0	−·73476 985	+2·18209 15	+1	−·93154 984
−1	+·42803 606	+ ·01934 438	+ ·01739 126	+·02782 838
−2	+·00387 413	+ ·00016 590	+ ·00015 057	+·00024 067
−3	+·00003 326	+ ·00000 142	+ ·00000 129	+·00000 206
−4	+·00000 029	+ ·00000 001	+ ·00000 001	+·00000 002
Sum	−·30602 318	+2·03549 49	+1·02059 366	−·90857 983

q=+1·51415 29.

71. In the terms of the third order the above process has been actually used. In those of higher orders a slight abbreviation of the work will be made by the following change.

Put

$$A=\tfrac{3}{4}\frac{\kappa u_0\zeta^{-1}}{\rho_0^{\,3}}A_1, \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad (10).$$

and, instead of u_2, s_2, s_3, s_4 use each of these series multiplied by $3\kappa u_0\zeta^{-1}/4\rho_0^{\,3}$. The latter four new series can be obtained once for all.

The advantage gained from this change will be seen by looking at equation (17) of Chap. I. The terms A consist of two parts. The first is the part due to the first line of the second member of the equation; the terms in this line are small, being due to δc and ϖ, and their multiplication by $4\rho_0^{\,3}/3\kappa u_0\zeta^{-1}$ to get the corresponding part of A_1 will be short. The second is the part due to the succeeding lines of the equation; these always form the principal part of A, and they all have the above-mentioned factor.

Section (iii). *The Terms of Long Period.*

72. Small divisors arise when the period of any term approximates to that of the principal elliptic term or when it is long. In the former case, the small divisor arises in the first and second terms of (8), being due to the

operators $(D\pm c_0)^{-1}$. In the latter case, as was pointed out (Chap. I., § 28 (*c*)), the *square* of the divisor occurs, but this difficulty may be avoided by the use of the homogeneous equation (21). I shall now show how the latter equation may be adapted when the methods of the last section are used.

The long-period small divisor arises only in V, T, owing to the operator D^{-1}. But (omitting the suffix λ for brevity)

$$\mathrm{T}=D^{-1}(\mathrm{qV}-\mathrm{W}),\ \mathrm{V}=D^{-1}(A\mathrm{s}_3+\bar{A}\mathrm{u}_3),$$

so that the *square* of the small divisor arises in the part

$$D^{-1}(\mathrm{qV})=\mathrm{q}D^{-2}(A\mathrm{s}_3+\bar{A}\mathrm{u}_3)$$

of T. This is therefore the term to be considered.

We have

$$\begin{aligned} D\mathrm{V}=A\mathrm{s}_3+\bar{A}\mathrm{u}_3&=\tfrac{1}{\mathrm{a}}(A\zeta Ds_0+\bar{A}\zeta^{-1}Du_0)\\ &=\tfrac{1}{\mathrm{a}}\{(\bar{A}\zeta^{-1}u_0-A\zeta s_0)+\bar{A}\zeta^{-1}(Du_0-u_0)+A\zeta(Ds_0+s_0)\} \quad \ldots \quad (11)\end{aligned}$$

But by equation (21), Chap. I.,

$$u_0D^2s_\lambda+u_\lambda D^2s_0-s_0D^2u_\lambda-s_\lambda D^2u_0-2\mathrm{m}D\,(s_0u_\lambda+u_0s_\lambda)+3\mathrm{m}^2(u_0u_\lambda-s_0s_\lambda)=\mathrm{a}^2\lambda\Lambda',$$

where Λ' denotes all the terms of characteristic λ except those due to u_λ, s_λ, in the expansion of

$$D(sDu-uDs+2\mathrm{m}us)+\frac{3}{2}\mathrm{m}^2(s^2-u^2)+s\frac{\partial\Omega}{\partial s}-u\frac{\partial\Omega_1}{\partial u} \quad \ldots \quad \ldots \quad (12)$$

This equation may be written—

$$\begin{aligned} &u_0\left[(D-\mathrm{m})^2s_\lambda+\frac{1}{2}\left(\mathrm{m}^2+\frac{\kappa}{r_0^{\,3}}\right)s_\lambda+\frac{3}{2}\left(\mathrm{m}^2+\frac{\kappa s_0^{\,2}}{r_0^{\,5}}\right)u_\lambda\right]\\ &-s_0\left[(D+\mathrm{m})^2u_\lambda+\frac{1}{2}\left(\mathrm{m}^2+\frac{\kappa}{r_0^{\,3}}\right)u_\lambda+\frac{3}{2}\left(\mathrm{m}^2+\frac{\kappa u_0^{\,2}}{r_0^{\,5}}\right)s_\lambda\right]=\mathrm{a}^2\lambda\Lambda',\end{aligned}$$

or, using equation (24) of Chap. I.,

$$u_0\bar{A}\zeta^{-1}-s_0A\zeta=\mathrm{a}\Lambda'.$$

Whence, restoring the suffix λ,

$$\mathrm{V}_\lambda=D^{-1}\Lambda_\lambda'+\frac{1}{\mathrm{a}}D^{-1}\{\bar{A}\zeta^{-1}(Du_0-u_0)+A\zeta(Ds_0+s_0)\} \quad \ldots \quad \ldots \quad (13)$$

where $\mathrm{a}^2\lambda\, D^{-1}\Lambda_\lambda'$ will denote the *known* terms of characteristic λ, in

$$sDu-uDs+2\mathrm{m}us+D^{-1}\left\{\frac{3}{2}\mathrm{m}^2(s^2-u^2)+s\frac{\partial\Omega_1}{\partial s}-u\frac{\partial\Omega_1}{\partial u}\right\} \quad \ldots \quad \ldots \quad (14)$$

The portion of (14) under the operator D^{-1} contains the factor m^2 at least; hence the effect of the small divisor (which is never of an order higher than m^2) is neutralised in the first term of V. The same thing occurs with the remainder of the expression for V, owing to the fact that Du_0-u_0 and Ds_0+s_0 contain the factor m^2 at least. Hence V can be found to the same degree of accuracy as A, and the loss of accuracy in u_λ is limited to that due to the *first* power of the small divisor.

In using this we first find A as usual, and with it calculate the coefficient of the particular power of ζ in the second part of (13). The terms containing this power of ζ for the given value of λ are then chosen out of the expression (14), and thence V is obtained to the required accuracy for this power of ζ. The method of Section (ii) serves for all the other powers of ζ.

Section (iv). *The New Parts of the Motions of the Perigee and Node.*

73. The method of calculating $c_{\lambda/e}$ is explained in Chap. I., § 28 (b); when this quantity has been obtained A is completely known. Putting $\lambda_0 = \lambda_0'$ (§25) and omitting one of the equations for $i=0$, we can solve the linear equations by continued approximation. The omitted equation serves as a test. Either this or the method of Section (ii) somewhat modified, can be used, but it is more convenient to proceed in the following manner.

In § 28 (b), we have put $A=B+c_{\lambda/e}\, b$. Hence write the solution in the form

$$\left.\begin{aligned}\lambda_i &= {}_1\lambda_i+\lambda_0(f)_i+c_{\lambda/e}(c)_i,\\ \lambda'_i &= {}_1\lambda_i'+\lambda_0(f')_i+c_{\lambda/e}(c')_i,\end{aligned}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

except for $i=0$.

Here ${}_1\lambda_i$, ${}_1\lambda_i'$, are obtained by solving equation (26) of Chap. I. with $A_i=B_i$, $A_i'=B_i'$, $\lambda_0=0=\lambda_0'$.

The terms $(f)_i\,\lambda_0$, $(f')_i\,\lambda_0$ are obtained by solving the same equation with $A_i=0=A_i'$, $\lambda_0=\lambda_0'$ (§25), in terms of λ_0. (If, instead of $\lambda_0=\lambda_0'$, we had put $\lambda_0-\lambda_0'=1$, we should have found ε_i, ε_i'.)

The terms $(c)_i$, $(c')_i$ are obtained by solving with $A_i=b_i$, $A_i'=b_i'$, $\lambda_0=0=\lambda'_0$.

In all three cases the equations for $i=0$ are omitted.

We now substitute the values of λ_i, λ'_i, thus obtained, in the two equations for $i=0$ and obtain two equations of the form

$$\left.\begin{array}{l} {}_1\lambda_0+\lambda_0(f_0)+\mathrm{c}_{\lambda/\mathrm{e}}\,(c_0)=0 \\ {}_1\lambda'_0+\lambda_0(f_0')+\mathrm{c}_{\lambda/\mathrm{e}}\,(c_0')=0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

in which everything is known except λ_0. If the work be eorrect these should give the same value of λ_0, and thence, substituting in (15), the values of λ_i, λ'_i.

The coefficients $(f)_i$, $(f')_i$, $(c)_i$, $(c')_i$ being independent of λ, are found once for all ; they are given below. The advantage of this procedure arises from the omission of the equations for $i=0$ in the solution of the linear equations : the approximations are rapid.

The same process is used for the nodal motion with the z-equation. But as we here put $\lambda_0=0=\lambda_0'$ (§ 26), the equations (15), (16) reduce to

$$\lambda_i={}_1\lambda_i+\mathrm{g}_{\lambda/\mathrm{k}}(g)_i,\ (\text{except } i=0) \quad \ldots \quad \ldots \quad \ldots \quad (15)'$$

$$ {}_1\lambda_0+\mathrm{g}_{\lambda/\mathrm{k}}(g_0)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)'$$

The equation (16)′ serves then as a test. In all coefficients arising in the z-equation we have $q_i=-q'_{-i}$.

74. The following are the numerical results required for the method of this Seetion :—

i.	$(f)_i$.	$(f')_i$.
3	−·00000 009	−·00000 039
2	−·00001 269	−·00006 201
1	−·00213 186	−·01412 29
0		
−1	+·01261 31	−·00293 224
−2	+·00002 655	−·00000 071
−3	−·00000 001	
Sum ...	+·01049 50	−·01711 83

$(f_0)=+6{\cdot}97857\ 932$
$(f'_0)=+2{\cdot}34613\ 680$
$(c_0)=+1{\cdot}07940\ 266$
$(c'_0)=+\ {\cdot}02014\ 438$

i.	$(c)_i$.	$(c')_i$.	$(g)_i$.
4		− ·00000 007	
3	− ·00000 002	− ·00000 936	− ·00000 001
2	− ·00000 348	− ·00135 001	− ·00000 298
1	− ·00071 843	− ·22311 94	− ·00111 579
0			
−1	+ ·71453 80	+ ·00038 928	+ ·19835 12
−2	+ ·00001 970	+ ·00000 037	+ ·00030 486
−3	− ·00000 066		+ ·00000 118
−4	− ·00000 001		+ ·00000 001
Sum ...	+ ·71383 51	− ·22408 92	+ ·19753 85

Section (v). *Values of A*, $u_\lambda\zeta^{-1}/a\lambda$, $\iota z_\lambda/a\lambda$.

75. The following tables show the characteristics, arguments, and types of coefficients of the terms of the third order according to the scheme adopted in Chap. I., Section (iv). The numerical results are given below in the same order, and will be found in the §§ given in the first columns.

§	λ	Arguments.	Types of Coefficients in $u_\lambda\zeta^{-1}/a\lambda$.
76	e^3	$2i\pm3c$; $2i\pm c$	(ϵ'), (ϵ'^3) ; $(\epsilon^2\epsilon')$, $(\epsilon\epsilon'^2)$
77	e^2e'	$2i\pm(2c+m)$; $2i\pm(2c-m)$; $2i\pm m$	$(\epsilon^2\eta)$, $(\epsilon'^2\eta')$; $(\epsilon^2\eta')$, $(\epsilon'^2\eta)$; $(\epsilon\epsilon'\eta)$, $(\epsilon\epsilon'\eta')$
78	ee'^2	$2i\pm(c+2m)$; $2i\pm(c-2m)$; $2i\pm c$	$(\epsilon\eta^2)$, $(\epsilon'\eta'^2)$; $(\epsilon\eta'^2)$, $(\epsilon'\eta^2)$; $(\epsilon\eta\eta')$, $(\epsilon'\eta\eta')$
79	e'^3	$2i\pm3m$; $2i\pm m$	(η^3), (η'^3) ; $(\eta^2\eta')$, $(\eta\eta'^2)$
80	ek^2	$2i\pm(c+2g)$; $2i\pm(c-2g)$; $2i\pm c$	(ϵk^2), $(\epsilon'k'^2)$; $(\epsilon k'^2)$, $(\epsilon'k^2)$; $(\epsilon kk')$, $(\epsilon'kk')$
81	$e'k^2$	$2i\pm(m+2g)$; $2i\pm(m-2g)$; $2i\pm m$	(ηk^2), $(\eta'k'^2)$; $(\eta k'^2)$, $(\eta'k^2)$; $(\eta kk')$, $(\eta'kk')$
82	$e^2\alpha$	$2i_1\pm2c$; $2i_1$	$(\epsilon^2\alpha)$, $(\epsilon'^2\alpha)$; $(\epsilon\epsilon'\alpha)$
83	$ee'\alpha$	$2i_1\pm(c+m)$; $2i_1\pm(c-m)$	$(\epsilon\eta\alpha)$, $(\epsilon'\eta'\alpha)$; $(\epsilon\eta'\alpha)$, $(\epsilon'\eta\alpha)$
84	$e'^2\alpha$	$2i_1\pm2m$; $2i_1$	$(\eta^2\alpha)$, $(\eta'^2\alpha)$; $(\eta\eta'\alpha)$
85	$k^2\alpha$	$2i_1\pm2g$; $2i_1$	$(k^2\alpha)$, $(k'^2\alpha)$; $(kk'\alpha)$
86	$e\alpha^2$	$2i\pm c$	$(\epsilon\alpha^2)$, $(\epsilon'\alpha^2)$
87	$e'\alpha^2$	$2i\pm m$	$(\eta\alpha^2)$, $(\eta'\alpha^2)$
88	α^3	$2i_1$	(α^3)

§	λ	Arguments.	Types of Coefficients in $\iota z_\lambda/a\lambda$.
89	ke^2	$\pm(2i+g\pm 2c)$; $\pm(2i+g)$	$\pm(k\epsilon^2)$, $\pm(k\epsilon'^2)$; $\pm(k\epsilon\epsilon')$
90	kee'	$\pm\{2i+g\pm(c+m)\}$; $\pm\{2i+g\pm(c-m)\}$	$\pm(k\epsilon\eta)$, $\pm(k\epsilon'\eta')$; $\pm(k\epsilon\eta')$, $\pm(k\epsilon'\eta)$
91	ke'^2	$\pm(2i+g\pm 2m)$; $\pm(2i+g)$	$\pm(k\eta^2)$, $\pm(k\eta'^2)$; $\pm(k\eta\eta')$
92	k^3	$\pm(2i+3g)$; $\pm(2i+g)$	$\pm(k^3)$; $\pm(k^2k')$
93	$ke\alpha$	$\pm(2i_1+g\pm c)$	$\pm(k\epsilon\alpha)$, $\pm(k\epsilon'\alpha)$
94	$ke'\alpha$	$\pm(2i_1+g\pm m)$	$\pm(k\eta\alpha)$, $\pm(k\eta'\alpha)$
95	$k\alpha^2$	$\pm(2i+g)$	$\pm(k\alpha^2)$

The following long-period terms have been obtained with the required accuracy by the method of Section (iii) :—

Arguments.	Coefficients.
$\pm(-2+2c+m)$,	$(\epsilon^2\eta)_{-1}$, $(\epsilon'^2\eta')_1$
$\pm(-2+2c-m)$,	$(\epsilon^2\eta')_{-1}$, $(\epsilon'^2\eta)_1$
$\pm m$,	$(\epsilon\epsilon'\eta)_0$, $(\epsilon\epsilon'\eta')_0$
$\pm(-1+c-m)$,	$(\epsilon\eta'\alpha)_{-\frac{1}{2}}$, $(\epsilon'\eta\alpha)_{\frac{1}{2}}$

The values of V_λ for these terms, obtained as shown in Section (ii), agreed, as far as they went, with the values obtained by Section (iii); this agreement furnished a valuable test.

Equation (8) of Chap. I. was used to obtain $(k\eta'^2)_{-1}$ with sufficient accuracy.

Preliminary values of the parts of the motion of the perigee and node having e^2, e'^2, k^2, as factors were given in an Appendix at the end of Chap. IV. The values found below differ slightly from these. This is partly due to the fact that they have been re-calculated by a different and more accurate method. In one case—that of g_{e^2}—an error was found in one of the final steps of the early calculations (where no test equation had been computed); this induced an error in c_{k^2} which was deduced from it by using the connecting relation which I gave in a paper, "Investigations in the Lunar Theory." * These two quantities having been re-calculated independently, and the values satisfying the relation just mentioned, they may be accepted as final.

* *Amer. Jour. Math.*, Vol. xvii. p. 349.

The numerical results now follow. The values of A_λ for the terms arising in u are given in two parts—those arising from the expansion of $\kappa u\zeta^{-1}/r^3$, denoted by K_λ, and those arising from $-\frac{\partial\Omega_1}{\partial s}\zeta^{-1}$, denoted by Ω_λ. Then

$$A_\lambda = K_\lambda + \Omega_\lambda,$$

except for the exponents $2i\pm c$, $2i\pm g$, where

$$B_\lambda = K_\lambda + \Omega_\lambda.$$

The numbers are the coefficients corresponding to the power of ζ (that is, the argument) which is placed at the head of each pair of columns. The separation of A_λ, B_λ in z is unnecessary, as the parts arising from Ω require but little calculation.

The suffix of i_1 is omitted in the tables. Further details concerning the results will be found in Chaps. I.-IV.

76. *Characteristic* e^3.

$\Omega_{e^3} = 0$. Values of

	$K_{e^3} = A_{e^3}$.		$K_{e^3} = B_{e^3}$.	
i.	$2i + 3c$.	$2i - 3c$.	$2i + c$.	$2i - c$.
5		+·00000 615	+·00000 015	+·00000 175
4	+·00000 025	+·00025 037	+·00000 883	+·00009 139
3	+·00001 501	+·00618 376	+·00046 189	+·00371 182
2	+·00079 199	+·00932 740	+·01879 136	+·08978 705
1	+·03251 1864	−·05285 4356	+·45337 5630	+·00093 0745
0	+·79860 3673	+·09945 4882	−·29234 3773	−·09815 4555
−1	−·18057 2435	+·00264 8901	−·00031 9004	+·05851 8714
−2	+·00415 328	+·00004 931	+·01207 313	+·00155 388
−3	+·00086 991	+·00000 074	+·00031 268	+·00002 902
−4	+·00002 152		+·00000 579	+·00000 048
−5	+·00000 040		+·00000 007	
Sum	+·65639 546	+·06506 716	+·19236 675	+·05647 029

Values of

i.	$(\epsilon^3)_i$.	$(\epsilon'^3)_i$.	$(\epsilon^2\epsilon')_i$.	$(\epsilon\epsilon'^2)_i$.
5		+·00000 010		+·00000 002
4		+·00000 701	+·00000 009	+·00000 138
3	+·00000 014	+·00038 889	+·00000 678	+·00009 907
2	+·00001 122	+·00176 65	+·00048 105	+·00528 276
1	+·00079 516	−·00218 478	+·02520 232	−·00568 84
0	+·04147 214	+·00522 993	−·04231 894	−·04231 894
−1	−·03023 295	+·00004 365	+·01685 68	+·00328 278
−2	+·00184 26	+·00000 052	+·00068 577	+·00002 537
−3	+·00004 509	+·00000 001	+·00000 497	+·00000 031
−4	+·00000 032		+·00000 006	
−5				
Sum	+·01393 37	+·00525 18	+·00091 89	−·03931 56

$c_{e^3} = + \cdot 00268 \;\; 571$

77. *Characteristic* e^2e'.

Values of $K_{e^2e'} + \Omega_{e^2e'}$.

i.	$2i+2c+m$.		$2i-2c-m$.	
5			+·00000 711	
4	− ·00000 044		+·00033 841	
3	− ·00002 526		+·01192 768	−·00000 024
2	− ·00120 655	−·00000 006	+·21675 309	−·00002 100
1	− ·04218 737	−·00000 5532	−·21126 011	−·00007 0187
0	− ·74778 991	−·00046 3158	−·13819 518	−·00047 5411
−1	− ·27045 446	−·00077 1829	−·00445 874	+·00046 0625
−2	+ ·04960 312	−·00054 348	−·00009 192	+·00000 551
−3	+ ·00141 627	−·00014 708	−·00000 152	+·00000 006
−4	+ ·00002 806	−·00000 166	−·00000 003	
−5	+ ·00000 046	−·00000 002		
Sum	−1·01061 608	−·00193 282	−·12498 121	−·00010 064

i.	$2i+2c-m$.		$2i-2c+m$.	
5	+ ·00000 004		−·00000 108	
4	+ ·00000 249		−·00005 063	
3	+ ·00013 421		−·00181 981	−·00000 024
2	+ ·00580 739	−·00000 006	−·03489 248	−·00002 107
1	+ ·16168 213	−·00000 5509	−·01655 646	−·00012 2352
0	+ ·91682 513	−·00046 0625	+·18214 920	+·00208 0639
−1	− ·02119 681	+·00047 5411	+·01811 441	−·00322 6913
−2	− ·00724 120	+·00007 019	+·00046 244	−·00003 856
−3	− ·00020 193	+·00002 100	+·00000 837	−·00000 040
−4	− ·00000 395	+·00000 024	+·00000 013	
−5	− ·00000 009			
Sum	+1·05580 741	+·00010 065	+·14741 409	−·00132 890

Characteristic e^2e'.

Values of $K_{e^2e'} + \varpi_{e^2e'}$.

i.	$2i+m$.		$2i-m$.	
5	−·00000 020		+ ·00000 109	
4	−·00001 021		+ ·00006 185	
3	−·00049 120	−·00000 002	+ ·00284 107	−·00000 002
2	−·01737 829	−·00000 227	+ ·09173 506	−·00000 226
1	−·32001 822	−·00019 2836	+1·28586 471	−·00019 1967
0	−·11424 811	+·00014 0471	− ·10877 253	+·00072 5730
−1	+·27689 127	+·00449 4856	− ·06274 527	−·00072 5730
−2	+·01060 170	−·00134 464	− ·00188 121	+.00019 197
−3	+·00023 060	−·00001 580	− ·00003 788	+·00000 226
−4	+·00000 392	−·00000 016	− ·00000 057	+·00000 002
−5	+·00000 006		− ·00000 001	
Sum	−·16441 868	+·00307 960	+1·20706 631	0

Values of

i.	$(\epsilon^2\eta)_i$.	$(\epsilon'^2\eta')_i$.	$(\epsilon^2\eta')_i$.	$(\epsilon'^2\eta)_i$.
5		+·00000 008		−·00000 001
4		+·00000 691	+·00000 002	−·00000 099
3	−·00000 029	+·00047 850	+·00000 156	−·00006 734
2	−·00002 191	+·02396 981	+·00011 095	−·00327 305
1	−·00143 289	+·03768 72	+·00590 206	+·02902 28
0	−·06288 748	−·01411 942	+·08337 053	+·02431 890
−1	−·14741 25	−·00006 809	−·03817 68	+·00036 494
−2	+·00695 047	−·00000 112	−·00113 005	+·00000 631
−3	+·00001 714	−·00000 002	−·00000 207	+·00000 007
−4	+·00000 036		−·00000 004	
−5				
Sum	−·20478 71	+·04795 39	+·05007 62	+·05037 16

Characteristic e^2e'.

Values of

i.	$(\epsilon\epsilon'\eta)_i$.	$(\epsilon\epsilon'\eta')_i$.
5		+ ·00000 001
4	− ·00000 012	+ ·00000 076
3	− ·00000 929	+ ·00005 597
2	− ·00062 354	+ ·00347 999
1	− ·02957 776	+ ·12638 533
0	− ·19519 75	+ ·14774 72
−1	+ ·04554 004	− ·00648 982
−2	+ ·00017 353	− ·00002 504
−3	+ ·00000 307	− ·00000 046
−4	+ ·00000 003	− ·00000 001
−5		
Sum ...	− ·17969 15	+ ·27115 39

78. *Characteristic* ee'^2.

Values of $K_{ee'^2} + \aleph_{ee'^2}$.

i.	$2i+c+2m$.		$2i-c-2m$.	
5			+ ·00000 241	
4	+ ·00000 017		+ ·00012 651	
3	+ ·00000 873		+ ·00513 044	− ·00000 014
2	+ ·00029 198	+ ·00000 001	+ ·12386 659	− ·00001 320
1	+ ·00371 387	− ·00000 380	+ ·09957 946	− ·00120 073
0	− ·10101 661	− ·00138 503	− ·03198 329	+ ·00293 329
−1	+ ·05016 537	+ ·06034 441	+ ·00053 064	− ·00045 811
−2	+ ·02162 607	− ·01016 166	+ ·00002 797	− ·00000 705
−3	+ ·00056 850	− ·00010 231	+ ·00000 062	− ·00000 008
−4	+ ·00001 033	− ·00000 099	+ ·00000 001	
−5	+ ·00000 018	− ·00000 001		
Sum .	− ·02463 141	+ ·04869 062	+ ·19728 136	+ ·00125 398

Characteristic ee'^2.

Values of $K_{ee'^2} + \varpi_{ee'^2}$.

i.	$2i+c-2m$.		$2i-c+2m$.	
5	+ ·00000 016		+ ·00000 006	
4	+ ·00001 003		+ ·00000 193	
3	+ ·00049 475		+ ·00006 588	
2	+ ·01857 417	− ·00000 053	+ ·00106 031	− ·00000 069
1	+ ·38487 310	− ·00004 615	− ·01398 065	− ·00028 638
0	+ ·22941 999	− ·00247 794	+ ·08802 983	+ ·01764 269
−1	− ·00540 404	− ·00018 882	+ ·06570 344	− ·02534 819
−2	+ ·00015 577	− ·00011 935	+ ·00202 853	− ·00037 010
−3	+ ·00000 637	− ·00000 158	+ ·00003 953	− ·00000 403
−4	+ ·00000 014	− ·00000 002	+ ·00000 063	− ·00000 004
−5			+ ·00000 001	
Sum	+ ·62813 044	− ·00283 439	+ ·14294 950	− ·00836 674

i.	$2i+c$.		$2i-c$.	
5	− ·00000 005		− ·00000 073	
4	− ·00000 310		− ·00003 719	
3	− ·00016 037		− ·00153 051	− ·00000 011
2	− ·00628 749	− ·00000 043	− ·03870 578	− ·00001 092
1	− ·14287 981	− ·00003 572	− ·14727 365	− ·00093 229
0	− ·17975 583	− ·00139 125	− ·06087 008	+ ·01360 110
−1	− ·05798 850	− ·00761 209	− ·02253 804	+ ·01000 455
−2	− ·00621 038	+ ·00299 069	− ·00064 816	+ ·00012 062
−3	− ·00015 985	+ ·00002 956	− ·00001 220	+ ·00000 126
−4	− ·00000 283	+ ·00000 029	− ·00000 020	+ ·00000 001
−5	− ·00000 005			
Sum ...	− ·39344 826	− ·00601 895	− ·27161 654	+ ·02278 422

Characteristic ee'^2.

Values of

i.	$(\epsilon\eta^2)_i$.	$(\epsilon'\eta'^2)_i$.	$(\epsilon\eta'^2)_i$.	$(\epsilon'\eta^2)_i$.
4		+ ·00000 210	+ ·00000 010	
3	+ ·00000 011	+ ·00015 848	+ ·00000 759	− ·00000 227
2	+ ·00000 579	+ ·00994 322	+ ·00050 707	− ·00051 717
1	− ·00000 203	+ ·36932 90	+ ·02381 252	− ·10112 91
0	− ·05024 49	+ ·10676 52	+ ·14026 79	− ·23325 20
−1	− ·78467 70	− ·00001 322	+ ·29493 28	− ·00039 221
−2	− ·00147 012	+ ·00000 026	+ ·00003 655	+ ·00003 448
−3	+ ·00000 949	+ ·00000 001	− ·00000 018	+ ·00000 047
−4	+ ·00000 012			
Sum ...	− ·83637 85	+ ·48618 51	+ ·45956 44	− ·33525 78

i.	$(\epsilon\eta\eta')_i$.	$(\epsilon'\eta\eta')_i$.
4	− ·00000 003	− ·00000 059
3	− ·00000 236	− ·00004 352
2	− ·00016 176	− ·00264 878
1	− ·00797 920	− ·08755 90
0	− ·02023 661	− ·02023 661
−1	+ ·14228 77	+ ·00040 089
−2	+ ·00037 254	− ·00000 957
−3	− ·00000 231	− ·00000 014
−4	− ·00000 003	
Sum ...	+ ·11427 79	− ·11009 73

$c_{e'^2} = -·03465\ 60$

79. *Characteristic e'^3.*

Values of $K_{e'^3} + \mathfrak{S}_{e'^3}$.

i.	$2i+3$m.		$2i-3$m.	
5			+·00000 026	
4	−·00000 001		+·00001 439	
3	+·00000 003		+·00063 617	−·00000 001
2	+·00002 943	−·00000 001	+·01785 243	−·00000 180
1	+·00165 270	−·00000 866	+·04256 833	−·00017 494
0	+·02364 0522	−·00875 9209	+·01902 8519	−·01242 3782
−1	+·00827 230	−·18242 850	+·00027 208	−·00041 049
−2	+·00292 678	−·00159 328	+·00000 321	−·00000 047
−3	+·00006 929	−·00001 486		+·00000 001
−4	+·00000 116	−·00000 014		
−5	+·00000 002			
Sum	+·03659 222	−·19280 466	+·08037 539	−·01301 148

i.	$2i+$m.		$2i-$m.	
5	+·00000 001		−·00000 010	
4	+·00000 066		−·00000 629	
3	+·00002 869		−·00028 158	−·00000 001
2	+·00072 679	−·00000 001	−·00825 586	−·00000 123
1	+·00088 954	−·00002 564	−·04906 880	−·00011 396
0	−·02263 5731	−·00385 1214	−·02019 3073	−·00275 2931
−1	−·00919 324	+·08674 561	+·00009 023	−·00087 633
−2	−·00124 392	+·00067 905	+·00010 367	−·00005 386
−3	−·00002 899	+·00000 633	+·00000 278	−·00000 064
−4	−·00000 051	+·00000 005	+·00000 002	
−5				
Sum	−·03145 670	+·08355 418	−·07760 900	−·00379 896

Characteristic e'^3.

Values of

i.	$(\eta^3)_i$.	$(\eta'^3)_i$.	$(\eta^2\eta')_i$.	$(\eta\eta'^2)_i$.
4		+·00000 021	+·00000 001	−·00000 008
3		+·00001 620	+·00000 056	−·00000 655
2	−·00000 004	+·00113 404	+·00002 762	−·00044 497
1	−·00003 123	+·05909 118	+·00021 277	−·02145 800
0	−·04032 334	+·06189 440	−·00900 66	−·00500 46
−1	−·26598 707	−·00002 034	+·08932 148	−·00064 950
−2	+·00004 936		−·00001 321	+·00000 035
−3	+·00000 132		−·00000 049	+·00000 003
−4	+·00000 001			
Sum	−·30629 099	+·12211 569	+·08054 21	−·02756 33

80. *Characteristic* ek^2.

$\mathfrak{B}_{ek^2}=0$. Values of $K_{ek^2}=A_{ek^2}$.

i.	$2i+c+2g$.	$2i-c-2g$.	$2i+c-2g$.	$2i-c+2g$.
5		+ ·00000 043	+·00000 002	+ ·00000 001
4	+·00000 004	+ ·00002 579	+·00000 223	+ ·00000 066
3	+·00000 333	+ ·00117 070	+·00013 299	+ ·00004 388
2	+·00022 701	+ ·02820 996	+·00615 432	+ ·00236 501
1	+·01269 4440	− ·13388 3190	+·15651 5020	+ ·07579 4392
0	+·46462 6850	+1·31574 3309	−·43104 1098	−1·33711 7389
−1	−·45325 8160	+ ·02784 3049	+·25309 9644	+ ·05938 0541
−2	+·01230 676	+ ·00041 405	+·00545 699	+ ·00187 632
−3	+·00036 210	+ ·00000 526	+·00008 172	+ ·00003 410
−4	+·00000 667	+ ·00000 006	+·00000 106	+ ·00000 049
−5	+·00000 009		+·00000 001	
Sum	+·03696 913	+1·23952 942	−·00959 709	−1·19762 199

Characteristic ek².

$\aleph_{ek^2}=0$. Values of $K_{ek^2}=B_{ek^2}$.

i.	$2i+c$.	$2i-c$.
5		−·00000 005
4	−·00000 025	−·00000 313
3	−·00001 523	−·00015 379
2	−·00072 658	−·00524 052
1	−·02123 4090	−·09199 9007
0	−·03507 4881	−·02732 4124
−1	−·04932 6078	+·06866 6563
−2	+·01296 733	+·00145 824
−3	+·00028 345	+·00002 175
−4	+·00000 425	+·00000 028
−5	+·00000 007	
Sum ...	−·09312 201	−·05457 379

Values of

i.	$(\epsilon k^2)_i$.	$(\epsilon' k'^2)_i$.	$(\epsilon k'^2)_i$.	$(\epsilon' k^2)_i$.
4		+·00000 053	+·00000 002	
3		+·00004 989	+·00000 085	−·00000 008
2	+·00000 027	+·00385 36	−·00001 633	−·00001 355
1	+·00002 256	−·01711 531	−·02889 967	−·00235 066
0	+·00138 166	+·24987 870	+·64704 18	−·46343 59
−1	−·07120 080	+·00146 418	+·05538 425	+·17077 937
−2	+·00889 21	+·00001 008	+·00030 609	+·00029 727
−3	+·00005 331	+·00000 007	+·00000 207	+·00000 136
−4	+·00000 030		+·00000 002	+·00000 002
Sum	−·06085 06	+·23814 17	+·67381 91	−·29472 22

Characteristic ek².

Values of

i.	$(\epsilon k k')_i$.	$(\epsilon' k k')_i$.
4		−·00000 012
3	−·00000 041	−·00001 274
2	−·00003 797	−·00130 683
1	−·00279 137	−·13063 320
0	−·01298 640	−·01298 640
−1	+·29872 990	+·01621 815
−2	+·00369 243	+·00008 801
−3	+·00001 816	+·00000 059
−4	+·00000 012	
Sum ...	+·28662 446	−·12863 254

$c_{k^2} = +·05385\ 595$

81. *Characteristic* e′k².

Values of $K_{e'k^2} + \aleph_{e'k^2}$.

i.	$2i+m+2g$.		$2i-m-2g$.	
5			+·00000 015	
4	−·00000 001		+·00000 870	
3	−·00000 051		+·00043 519	−·00000 001
2	−·00001 838		+·01232 953	−·00000 099
1	+·00038 7658	−·00000 0271	−·24500 7602	−·00020 8242
0	+·09673 8490	−·00005 9900	+·09029 2445	−·00529 7319
−1	−·29235 4798	−·03343 2753	−·00832 9223	+·00000 0725
−2	+·00803 763	−·00148 971	−·00018 592	+·00000 004
−3	+·00017 185	−·00000 704	−·00000 278	
−4	+·00000 255	−·00000 005	−·00000 004	
−5	+·00000 003			
Sum	−·18703 549	−·03498 972	−·15045 955	−·00550 580

Characteristic $e'k^2$.

Values of $K_{e'k^2} + ☊_{e'k^2}$.

i.	2*i*+m−2g.		2*i*−m+2g.	
5	−·00000 002			
4	−·00000 210		+·00000 012	
3	−·00011 304	−·00000 001	+·00000 679	
2	−·00357 523	−·00000 105	+·00035 223	
1	+·10326 8572	−·00024 0255	+·01107 0715	−·00000 0040
0	−·12143 6400	−·00164 8838	−·11412 7077	−·00000 0725
−1	+·06401 3724	−·00006 4252	+·10878 3159	+·00529 7319
−2	+·00135 509	−·00000 051	−·00344 466	+·00020 824
−3	+·00002 016		−·00005 398	+·00000 099
−4	+·00000 028		−·00000 067	+·00000 001
−5				
Sum	+·04353 104	−·00195 492	+·00258 663	+·00550 579

i.	2*i*+m.		2*i*−m.	
5				
4	−·00000 037		+·00000 015	
3	−·00003 183		+·00003 380	
2	−·00228 182	+·00000 013	+·00379 855	+·00000 014
1	−·11219 5215	+·00001 8123	+·24359 2093	+·00002 1752
0	+·01112 0432	+·00210 9609	+·01085 3449	+·00530 5862
−1	+·24707 3790	+·03394 4779	−·11769 1816	−·00530 5862
−2	+·00642 043	+·00014 863	−·00193 573	−·00002 175
−3	+·00010 319	+·00000 095	−·00002 555	−·00000 014
−4	+·00000 141	+·00000 001	−·00000 031	
−5	+·00000 002			
Sum	+·15021 004	+·03622 223	+·13862 464	0

Characteristic $e'k^2$.

Values of

i.	$(\eta k^2)_i$.	$(\eta' k'^2)_i$.	$(\eta k'^2)_i$.	$(\eta' k^2)_i$.
4		+·00000 016	−·00000 002	
3		+·00001 505	−·00000 389	+·00000 001
2	−·00000 001	+·00143 501	−·00047 166	+·00000 116
1	+·00000 630	+·10274 226	−·07635 55	+·00008 516
0	+·00163 202	+·04278 164	−·08014 891	+·00269 177
−1	−·21446 825	−·00081 115	+·00666 731	+·12718 46
−2	+·00412 476	−·00000 673	+·00005 120	−·00186 746
−3	+·00001 413	−·00000 006	+·00000 039	−·00000 421
−4	+·00000 009			−·00000 004
Sum	−·20869 096	+·14615 618	−·15026 11	+·12809 10

i.	$(\eta k k')_i$.	$(\eta' k k')_i$.
4		−·00000 001
3	+·00000 045	−·00000 270
2	+·00004 861	−·00025 922
1	+·00488 835	−·02176 661
0	+·48307 75	−·52538 39
−1	+·24918 465	−·08400 495
−2	+·00065 513	−·00016 048
−3	+·00000 383	−·00000 081
−4	+·00000 003	
Sum ...	+·73785 86	−·63157 87

82. *Characteristic* $e^2\alpha$.

Values of $K_{e^2\alpha} + \aleph_{e^2\alpha}$.

$2i$.	$2i+2c$.		$2i-2c$.	
9	+ ·00000 001		− ·00001 077	
7	+ ·00000 074		− ·00082 168	− ·00000 035
5	− ·00008 103	− ·00000 009	− ·03612 906	− ·00002 740
3	− ·01077 050	− ·00000 799	− ·06573 304	+ ·00014 077
1	− 56564 675	− ·00061 101	+ ·18304 866	− ·00221 738
−1	+ ·57575 118	+ ·00081 292	− ·07199 098	+ ·00283 928
−3	− ·02847 660	− ·00807 552	− ·00053 429	− ·00306 906
−5	− ·00534 300	+ ·00064 919	+ ·00000 433	− ·00004 003
−7	− ·00006 515	− ·00013 766	+ ·00000 021	− ·00000 043
−9	− ·00000 042	− ·00000 173		
Sum	− ·03463 152	− ·00737 189	+ ·00783 338	− ·00237 460

$2i$.	$2i$.	
9	− ·00000 031	
7	− ·00007 051	− ·00000 004
5	− ·00628 907	− ·00000 328
3	− ·29381 090	− ·00025 250
1	− ·17424 233	+ ·00173 425
−1	− ·06079 808	− ·00562 393
−3	− ·04075 791	+ ·00820 455
−5	− ·00043 426	− ·00126 848
−7	− ·00000 129	− ·00001 642
−9	+ ·00000 003	− ·00000 018
Sum ...	− ·57640 463	+ ·00277 397

Characteristic $e^2\alpha$.

Values of

$2i$.	$(\epsilon^2\alpha)_i$.	$(\epsilon'^2\alpha)_i$.	$(\epsilon\epsilon'\alpha)_i$.
9		− ·00000 017	
7	+ ·00000 002	− ·00002 360	− ·00000 100
5	− ·00000 099	− ·00247 899	− ·00016 334
3	− ·00025 786	− ·06279 45	− ·01741 144
1	− ·03029 187	− ·07046 35	− ·10960 58
−1	+ ·12685 76	− ·00347 720	+ ·21027 9
−3	+ ·11551 33	− ·00016 375	− ·00049 006
−5	− ·00010 571	− ·00000 080	− 00007 754
−7	− ·00000 946	− ·00000 001	− ·00000 038
−9	− ·00000 006		
Sum ...	+ ·21170 50	− ·13940 25	+ ·08252 9

83. *Characteristic* $ee'\alpha$.

Values of $K_{ee'\alpha} + \text{ꝏ}_{ee'\alpha}$.

$2i$.	$2i+c+m$.		$2i-c-m$.	
9	+ ·00000 05		+ ·00000 19	
7	+ ·00003 67		− ·00008 15	− ·00000 021
5	+ ·00250 00	+ ·00000 004	− ·01113 14	− ·00001 929
3	+ ·13660 87	+ ·00000 161	− ·33312 49	− ·00150 657
1	+ 4·65374 56	− ·00038 845	− ·15457 19	+ ·01267 384
−1	− ·19743 34	+ ·00791 743	+ ·93078 13	+ ·00811 527
−3	− ·06923 50	+ ·09670 148	+ ·01512 50	+ ·00880 652
−5	− ·00082 91	− ·01012 338	+ ·00018 99	+ ·00009 282
−7	+ ·00001 15	− ·00011 510	+ ·00000 21	+ ·00000 091
−9	+ ·00000 04	− ·00000 117		+ ·00000 001
Sum	+ 4·52540 59	+ ·09399 246	+ ·44719 05	+ ·02816 330

Characteristic $ee'\alpha$.

Values of $K_{ee'\alpha} + \mathfrak{S}_{ee'\alpha}$.

$2i$.	$2i+c-m$.		$2i-c+m$.	
9	+ ·00000 06		+ ·00000 75	
7	+ ·00001 84	− ·00000 001	+ ·00050 84	+ ·00000 001
5	+ ·00022 61	− ·00000 083	+ ·02742 74	+ ·00000 061
3	− ·01956 28	− ·00007 155	+ ·90964 64	− ·00004 517
1	− ·37493 92	− ·00415 929	− ·80846 72	+ ·00545 089
−1	− ·79391 61	+ ·00386 559	− ·07537 94	+ ·01923 289
−3	+ ·19032 70	− ·01502 541	+ ·00025 71	− ·03283 987
−5	+ ·00311 05	+ ·00197 731	+ ·00010 73	− ·00044 785
−7	+ ·00003 94	+ ·00001 999	+ ·00000 28	− ·00000 485
−9	+ ·00000 04	+ ·00000 020	+ ·00000 01	− ·00000 005
Sum	− ·99469 57	− ·01339 400	+ ·05411 04	− ·00865 339

Values of

$2i$.	$(\epsilon\eta\alpha)_i$.	$(\epsilon'\eta'\alpha)_i$.	$(\epsilon\eta'\alpha)_i$.	$(\epsilon'\eta\alpha)_i$.
9				+ ·00000 01
7	+ ·00000 05	− ·00000 16	+ ·00000 02	+ ·00000 96
5	+ ·00004 40	− ·00051 09	+ ·00000 73	+ ·00099 47
3	+ ·00454 88	− ·05773 38	− ·00053 32	+ ·09020 60
1	+ ·40017 48	− ·13184 7	− ·04137 23	+ ·32607 7
−1	+ ·01876 3	+ ·13596 59	− ·78645 2	+ ·01623 91
−3	+ ·11064 27	+ ·00107 47	+ ·01255 04	− ·00345 75
−5	− ·00122 87	+ 00000 51	+ ·00025 44	− ·00001 33
−7	− ·00000 40	+ ·00000 01	+ ·00000 12	
−9				
Sum	+ ·53294 1	− ·05304 8	− ·81554 4	+ ·43005 6

84. *Characteristic* $e'^2\alpha$.

Values of $K_{e'^2\alpha} + \mathfrak{S}_{e'^2\alpha}$.

2i.	2i+2m.		2i−2m.	
9	−·00000 030		+·00000 124	
7	−·00001 921		+·00003 767	−·00000 002
5	−·00108 806	−·00000 011	−·00018 847	−·00000 286
3	−·04289 260	−·00001 784	−·08297 640	−·00026 318
1	−·49418 300	−·00443 201	−·07733 041	−·01694 340
−1	−·02683 885	+·00651 162	−·15682 286	+·00130 863
−3	−·01458 690	−·20106 340	−·00657 451	+·00001 217
−5	+·00020 815	−·00201 212	−·00010 781	−·00001 889
−7	+·00000 792	−·00001 913	−·00000 141	−·00000 025
−9	+·00000 015	−·00000 018	−·00000 002	
Sum	−·57939 270	−·20103 317	−·32396 298	−·01590 780

2i.	2i.	
9	+·00000 163	
7	+·00011 559	
5	+·00672 559	+·00000 012
3	+·25523 830	−·00001 306
1	+·50942 997	−·00515 710
−1	+·17913 588	−·00450 279
−3	+·04356 335	+·06839 549
−5	+·00066 997	+·00060 357
−7	+·00000 796	+·00000 560
−9	+·00000 007	+·00000 006
Sum ...	+·99488 831	+·05933 189

Characteristic $e'^2\alpha$.

Values of

$2i$.	$(\eta^2\alpha)_i$.	$(\eta'^2\alpha)_i$.	$(\eta\eta'\alpha)_i$.
7	−·00000 02	+·00000 11	+·00000 15
5	−·00002 75	+·00006 42	+·00015 08
3	−·00236 65	+·00181 74	+·01214 19
1	−·10866 00	+·07247 89	+·01287 7
−1	−·22746 69	+·06296 01	+·24960 2
−3	−·05932 75	−·00033 14	+·02080 95
−5	−·00010 54	−·00000 31	+·00004 58
−7	−·00000 03		+·00000 02
Sum ...	−·39795 43	+·13698 72	+·29562 9

85. *Characteristic* $k^2\alpha$.

Values of $K_{k^2\alpha} + \Omega_{k^2\alpha}$.

$2i$.	$2i+2g$.		$2i-2g$.	
9			−·00000 019	
7	−·00000 039		−·00001 583	−·00000 001
5	−·00003 127		−·00107 524	−·00000 139
3	−·00219 273	−·00000 028	−·02025 402	−·00023 440
1	−·10174 5302	−·00005 1640	+·07691 4545	−·00386 8083
−1	+·31567 8728	−·01402 7700	−·30052 2743	−·01234 2006
−3	−·00281 851	−·02319 391	−·00124 397	−·00008 150
−5	−·00082 686	−·00111 281	+·00001 569	−·00000 060
−7	−·00000 723	−·00000 681	+·00000 043	−·00000 001
−9	−·00000 007	−·00000 005		
Sum	+·20805 637	−·03839 320	−·24618 133	−·01652 800

Characteristic $k^2\alpha$.

Values of $K_{k^2\alpha} + \wp_{k^2\alpha}$.

$2i$.	$2i$.	
9	−·00000 013	
7	−·00000 700	
5	−·00034 902	+·00000 018
3	−·01038 934	+·00002 577
1	+·05314 0982	+·00381 1636
−1	+·04282 1667	+·02680 3012
−3	−·03191 551	+·02342 170
−5	−·00027 549	+·00014 068
−7	−·00000 145	+·00000 098
−9		+·00000 001
Sum ...	+·05302 471	+·05420 397

Values of

$2i$.	$(k^2\alpha)_i$.	$(k'^2\alpha)_i$.	$(kk'\alpha)_i$.
9		−·00000 002	
7		−·00000 229	+ ·00000 015
5	+·00000 011	−·00037 096	+ ·00002 509
3	+·00001 162	−·07251 89	+ ·00430 805
1	+·00131 775	−·43498 445	+ ·84123 81
−1	+·18884 734	−·06346 523	−2·33191 84
−3	+·16090 08	−·00005 398	− ·00244 631
−5	−·00041 866	+·00000 052	− ·00000 465
−7	−·00000 086		+ ·00000 001
−9	−·00000 001		
Sum ...	+·35065 81	−·57139 53	−1·48879 80

86. *Characteristic* ea^2.

Values of $K_{ea^2} + \aleph_{ea^2}$.

i.	$2i+c$.		$2i-c$.	
5			+·00000 05	
4	+·00000 24		+·00002 45	−·00000 02
3	+·00012 36	−·00000 01	+·00117 89	−·00000 22
2	+·00520 57	−·00001 19	+·04732 91	−·00023 56
1	+·16389 96	−·00121 83	+·07138 99	+·00490 20
0	−·13821 47	+·00035 22	−·04822 18	+·00736 36
−1	+·02858 03	+·01208 22	+·02221 98	+·00169 14
−2	+·00750 77	+·03393 88	+·00062 41	−·00922 16
−3	+·00012 52	−·00184 24	+·00001 36	−·00008 67
−4	+·00000 23	−·00001 63	+·00000 02	−·00000 08
−5	+·00000 01	−·00000 01		
Sum ...	+·06723 22	+·04328 41	+·09455 88	+·00440 99

Values of

i.	$(\epsilon\alpha^2)_i$.	$(\epsilon'\alpha^2)_i$.
4		+·00000 04
3	+·00000 22	+·00003 11
2	+·00015 71	+·00229 11
1	+·00899 53	+·10151 1
0	−·01666 28	−·01666 28
−1	−·22632 4	+·00183 95
−2	+·00943 38	−·00054 38
−3	−·00011 98	−·00000 21
−4	−·00000 04	
Sum ...	−·22451 9	+·08846 4

$c_{\alpha^2} = -·02212\ 6.$

87. *Characteristic* $e'\alpha^2$.

Values of $K_{e'\alpha^2} + \Omega_{e'\alpha^2}$.

i.	$2i+m$.		$2i-m$.	
5			+·00000 02	
4	+ ·00000 01		+·00000 56	
3	− ·00008 40	−·00000 01	+·00013 63	−·00000 07
2	− ·01320 15	−·00001 42	−·00223 69	−·00008 17
1	−1·06476 55	−·00261 78	+·04455 94	−·00852 49
0	+ ·23300 345	−·00545 494	+·20737 158	−·00450 490
−1	+ ·00948 95	−·01227 27	−·21793 97	+·00700 56
−2	− ·00075 36	−·08834 34	−·00061 52	+·01108 38
−3	+ ·00003 33	−·00070 66	+·00000 45	+·00005 95
−4	+ ·00000 08	−·00000 56	+·00000 01	+·00000 04
−5				
Sum ...	− ·83627 74	−·10941 53	+·03128 59	+·00503 71

Values of

i.	$(\eta\alpha^2)_i$.	$(\eta'\alpha^2)_i$.
4		+·00000 01
3	−·00000 22	+·00001 15
2	−·00051 33	+·00063 84
1	−·09673 62	+·00542 42
0	+·05209 5	+·07246 1
−1	−·01035 45	−·02641 18
−2	−·01043 59	+·00131 14
−3	−·00002 64	+·00000 27
−4	−·00000 02	
Sum ...	−·06597 4	+·05343 8

88. *Characteristic* α^3.

Values of

$K_{\alpha^3} + \mho_{\alpha^3}$.

2i.	2i.	
7	−·00001 2	
5	−·00094 2	−·00001 0
3	−·01711 5	−·00141 9
1	+·00371 9	−·00396 6
−1	+·00142 4	−·00726 8
−3	−·00256 8	−·01393 1
−5	−·00013 0	−·01353 6
−7		−·00008 9
Sum ...	−·01562 4	−·04021 9

2i.	$(\alpha^3)_i$.
7	+·00000 1
5	+·00001 5
3	−·00125 9
1	−·09620
−1	+·27025
−3	−·00349 9
−5	−·00085 8
−7	−·00000 2
Sum ...	+·16845

89. *Characteristic* k^3.

Values of

i.	A_{k^3}. 2i + 3g.	B_{k^3}. 2i + g.	$(\lambda^3)_i$.	$(\lambda^2\lambda')_i$.
5				
4		−·00000 010		−·00000 00014
3	+·00000 007	−·00000 744	+·00000 00009	−·00000 0164
2	+·00000 543	−·00049 643	+·00000 0111	−·00002 1515
1	+·00036 1540	−·02324 4118	+·00001 4145	−·00277 8178
0	+·01675 5677	−·01731 6970	+·00168 5259	0
−1	−·02777 8633	+·00363 2982	−·06974 485	−·01224 585
−2	+·00186 956	−·00042 284	−·00158 665	−·00007 8594
−3	−·00001 059	−·00000 958	−·00000 620	−·00000 0527
−4	−·00000 048	−·00000 014	−·00000 0039	−·00000 0004
−5	−·00000 001		−·00000 00002	
Sum ...	−·00879 744	−·03786 464	−·06963 822	−·01512 483

$g_{k^3} = -·00806\ 6255$.

90. *Characteristic* ke².

Values of

i.	A_{ke^2}.		B_{ke^2}.
	$2i+g+2c$.	$2i+g-2c$.	$2i+g$.
5		+·00000 036	+·00000 005
4	+·00000 012	+·00002 174	+·00000 360
3	+·00000 910	+·00106 751	+·00022 053
2	+·00056 581	+·03177 374	+·01086 145
1	+·02815 3088	−·00388 7463	+·32763 8723
0	+·86762 4162	+·00460 5675	+·00651 0052
−1	−·01571 9318	−·02414 0044	+·00267 1924
−2	+·00148 491	−·00079 507	−·00902 610
−3	−·00085 550	−·00001 611	−·00030 517
−4	−·00002 977	−·00000 025	−·00000 621
−5	−·00000 063		−·00000 010
Sum ...	+·88123 197	+·00863 009	+·33856 875

Values of

i.	$(k\epsilon^2)_i$.	$(k\epsilon'^2)_i$.	$(k\epsilon\epsilon')_i$.
5		+·00000 00048	+·00000 00004
4	+·00000 00010	+·00000 0487	+·00000 00463
3	+·00000 01131	+·00004 8331	+·00000 4718
2	+·00001 1586	+·00427 6263	+·00046 0150
1	+·00112 1329	+·01672 370	+·03928 1120
0	+·09379 8299	−·08141 572	0
−1	−·04400 679	−·00307 9003	−·00716 888
−2	−·00161 311	−·00003 4589	−·00124 5876
−3	−·00013 5500	−·00000 0350	−·00001 4005
−4	−·00000 1470	−·00000 00032	−·00000 0141
−5	−·00000 0015		−·00000 00013
Sum ...	+·04917 444	−·06348 088	+·03131 713

$g_{e^2} = +·00318\ 6183$.

91. *Characteristic* $\text{ke}\epsilon'$.

Values of $A_{\text{ke}\epsilon'}$.

i.	$2i+\text{g}+\text{c}+\text{m}$.	$2i+\text{g}-\text{c}-\text{m}$.	$2i+\text{g}+\text{c}-\text{m}$.	$2i+\text{g}-\text{c}+\text{m}$.
5		+·00000 021	+·00000 001	−·00000 003
4	−·00000 014	+·00001 399	+·00000 096	−·00000 205
3	−·00001 008	+·00076 253	+·00006 148	−·00011 203
2	−·00054 981	+·02945 645	+·00318 821	−·00437 856
1	−·02125 360	+·39674 479	+·10786 825	−·06183 438
0	−·28354 837	−·26071 788	+·33223 139	+·29819 178
−1	+·35289 056	+·03417 336	−·03919 252	−·05967 106
−2	−·01531 540	+·00126 893	+·00654 802	−·00435 546
−3	−·00102 615	+·00002 634	+·00025 093	−·00011 193
−4	−·00002 518	+·00000 041	+·00000 524	−·00000 203
−5	−·00000 045		+·00000 009	−·00000 003
Sum	+·03116 138	+·20172 913	+·41096 206	+·16772 422

Values of

i.	$(k\epsilon\eta)_i$.	$(k\epsilon'\eta')_i$.	$(k\epsilon\eta')_i$.	$(k\epsilon'\eta)_i$.
5		+·00000 00023	+·00000 00001	−·00000 00003
4	−·00000 00014	+·00000 02390	+·00000 00099	−·00000 00336
3	−·00000 01579	+·00002 3817	+·00000 09940	−·00000 3300
2	−·00001 5276	+·00220 1482	+·00009 2083	−·00029 9393
1	−·00132 5491	+·15623 236	+·00707 5760	−·02027 447
0	−·07509 456	+·22035 892	+·10625 739	−·25455 973
−1	−·31367 703	+·01194 449	+·03225 857	−·02563 651
−2	−·01000 189	+·00009 4212	+·00276 085	−·00033 4698
−3	−·00009 1842	+·00000 0822	+·00002 0390	−·00000 3552
−4	−·00000 0874	+·00000 00070	+·00000 0174	−·00000 00351
−5	−·00000 00083		+·00000 0002	−·00000 00003
Sum	−·40020 713	+·39085 635	+·14846 622	−·30111 172

92. *Characteristic* ke'^2.

Values of

i.	$A_{ke'^2}$.		$B_{ke'^2}$.
	$2i+g+2m$.	$2i+g-2m$.	$2i+g$.
5		+·00000 002	
4	+·00000 003	+·00000 220	−·00000 061
3	+·00000 163	+·00012 938	−·00003 696
2	+·00005 277	+·00598 953	−·00170 201
1	−·00018 037	+·15607 842	−·04382 8741
0	+·00147 449	−·00328 213	+·01075 1875
−1	+·15457 297	−·00057 354	−·04205 6780
−2	−·00623 771	−·00027 313	+·00314 314
−3	−·00023 118	−·00000 738	+·00009 501
−4	−·00000 469	−·00000 014	+·00000 183
−5	−·00000 009		+·00000 002
Sum ...	+·14944 785	+·15806 323	−·07363 323

Values of

i.	$(k\eta^2)_i$.	$(k\eta'^2)_i$.	$(k\eta\eta')_i$.
5		+·00000 00002	−·00000 00001
4	+·00000 00004	+·00000 00293	−·00000 00079
3	+·00000 0032	+·00000 2898	−·00000 0786
2	+·00000 1954	+·00027 1352	−·00007 1657
1	−·00002 8799	+·02119 6537	−·00525 9644
0	−·00455 928	+·00832 47	0
−1	−·25292 577	+·02434 304	+·12438 368
−2	−·00147 2986	+·00000 4232	+·00064 3737
−3	−·00001 3150	−·00000 0173	+·00000 5169
−4	−·00000 0120	−·00000 0002	+·00000 0046
−5	−·00000 00013		+·00000 00003
Sum ...	−·25899 812	+·05414 26	+·11970 054

$g_{e'^2} = +{\cdot}00564\ 6535$.

93. *Characteristic* kea.

Values of

$2i$.	A_{kea}. $2i+g+c$.	A_{kea}. $2i+g-c$.	$(k\epsilon\alpha)_i$.	$(k\epsilon'\alpha)_i$.
9	+·00000 001	+·00000 015	+·00000 00001	+·00000 00015
7	+·00000 105	−·00000 684	+·00000 00117	−·00000 0196
5	+·00001 277	−·00141 686	+·00000 0096	−·00006 6448
3	−·00397 241	−·10868 994	−·00017 8806	−·01376 9493
1	−·40314 298	+·00080 657	−·04580 8867	−·01190 80
−1	+·00752 301	−·01874 851	+·04850 79	+·09203 77
−3	−·03088 873	+·01689 021	+·06482 90	+·00233 1866
−5	+·00382 908	+·00023 291	+·00067 3777	+·00001 1563
−7	+·00006 094	+·00000 141	+·00000 3486	+·00000 0041
−9	+·00000 055	−·00000 002	+·00000 0017	−·00000 00003
Sum	−·42657 671	−·11093 092	+·06802 66	+·06863 70

94. *Characteristic* ke′α.

Values of

$2i$.	$A_{ke'\alpha}$. $2i+g+m$.	$A_{ke'\alpha}$. $2i+g-m$.	$(k\eta\alpha)_i$.	$(k\eta'\alpha)_i$.
9	+ ·00000 007	+ ·00000 015	+·00000 00008	+ ·00000 00015
7	+ ·00000 686	+ ·00000 797	+·00000 01232	+ ·00000 01283
5	+ ·00057 193	+ ·00028 744	+·00001 82526	+ ·00000 8209
3	+ ·03963 137	+ ·00232 832	+·00285 0132	+ ·00013 6733
1	+1·80162 111	− ·00466 404	+·51292 301	− ·00839 428
−1	− ·05655 671	+1·79013 231	+·04334 669	−1·51928 973
−3	+ ·03667 959	− ·03946 234	+·01706 372	− ·02090 548
−5	+ ·00051 284	− ·00099 194	+·00005 4299	− ·00009 7841
−7	− ·00000 093	− ·00001 460	+·00000 0060	− ·00000 0584
−9	− ·00000 014	− ·00000 016	−·00000 00013	− ·00000 00037
Sum	+1·82246 599	+1·74762 311	+·57625 629	−1·54854 285

95. *Characteristic* kα^2.

Values of

i.	$B_{k\alpha^2}$. $2i+g$.
4	$+\cdot00000$ 07
3	$+\cdot00004$ 57
2	$+\cdot00228$ 18
1	$+\cdot07204$ 28
0	$+\cdot02497$ 59
-1	$+\cdot02728$ 33
-2	$-\cdot00073$ 96
-3	$-\cdot00003$ 63
-4	$-\cdot00000$ 08
Sum	$+\cdot12585$ 35

i.	$(k\alpha^2)$.
4	$+\cdot00000$ 0009
3	$+\cdot00000$ 098
2	$+\cdot00009$ 683
1	$+\cdot00862$ 452
0	0
-1	$-\cdot07777$ 0
-2	$-\cdot00023$ 555
-3	$-\cdot00000$ 214
-4	$-\cdot00000$ 002
Sum	$-\cdot06928$ 5

$$g_\alpha = +\cdot01110 \quad 58$$

(*To be continued.*)

Haverford College, Pa., U.S.A.:
1899 *January* 21.

Errata in Part I.

Page 45, line 11, *for* " a' " *read* " a'."
" 63, last line, *for* " $\frac{3}{2}$m^2 " *read* " $\frac{3}{2}$m$^2\zeta^{-2}$."
" 69, line 9, *for* " ; " *read* " , ".
" 75, line 4 from bottom, *for* " (. . .) " *read* " $(\ldots)^2$."
" 107, line 10, *for* " $-\cdot00585$ 014 " *read* " $-\cdot00585$ 0139."
" " " 11, *for* " $-\cdot01024$ 957 " *read* " $-\cdot01024$ 9560."
" " " 12, *for* " $+\cdot02515$ 958 " *read* " $+\cdot02515$ 9581."
" " " 17, *for* " $+\cdot00898$ 090 " *read* " $+\cdot00898$ 0914."
" 116, Appendix. See corrected values given above.

Theory of the Motion of the Moon; containing a New Calculation of the Expressions for the Coordinates of the Moon in Terms of the Time. By ERNEST W. BROWN, Sc.D., F.R.S.

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PART III. CHAPTER VI.

IN the first two parts of this Memoir, published under the same title in the *Memoirs of the Royal Astronomical Society* in 1897, 1899, the general theory and the numerical results, up to and inclusive of the third-order terms, have been given. This part contains the numerical results for the terms of the fourth order.

The methods adopted are in general the same as those used for the third-order terms. Instead of finding the values of u_λ/u_0 directly and then deducing those of u_λ, as in the third-order terms, I have found the values of u_λ directly. This change was found advisable when preparations were being made for the calculation of the fifth-order terms. It was seen that the non-homogeneous equations (17), (18) of Chap. I. would involve much more calculation than the homogeneous equations (6), (7), (8) of the same chapter, and the latter require the results for u_λ and not those for u_λ/u_0. In the former case u/r^3 has to be expanded to the fifth order, an enormous piece of work; in the latter case we only require the calculation of such expressions as u^2, uDs, &c., to the fifth order, and this has been so arranged as to require much less computation. The only other change from the methods of Chap. V. is that mentioned in § 71.

As in the earlier work all the assistance I have had in performing the computations has been rendered by Mr. IRA I. STERNER, A.M., and I take this opportunity of again expressing my obligations to him for the ability and accuracy with which he has conducted the work allotted to him.* I have also done a considerable amount of calculation myself, especially in the later portions of the work.

* The expense of making the computations necessary to obtain the results given below has been met by a grant from the Government Grant Fund of the Royal Society.

The following is the table of contents of Chap. V. :—

Section (i). A brief outline of the steps followed in the application of the general method to the terms of the fourth order.

Section (ii). The final numerical results for the series $\mathfrak{A}$, A, and for the coefficients of all terms of the fourth order in u, z.

Section (i). *Formulæ and Method of Procedure.*

96. The method employed is in general the same as that outlined in § 66. The products in the third line of equation (17) and in the second line of (18) of Chap. I. are formed by putting

$$u_\mu = u_1 + u_2 + u_3, \quad s_\mu = s_1 + s_2 + s_3, \quad z_\mu = z_1 + z_2 + z_3,$$

and choosing the parts of the fourth order. The products in the fourth and fifth lines of (17) and in the third line of (18) are similarly obtained with

$$u_\mu = u_1 + u_2, \quad s_\mu = s_1 + s_2, \quad z_\mu = z_1 + z_2.$$

The additional parts of (17), (18) in which we put

$$u_\mu = u_1, \quad s_\mu = s_1, \quad z_\mu = z_1,$$

are respectively

$$\frac{\kappa u_0 \zeta^{-1}}{\rho_0^3} \left\{ \frac{35}{128}\left(\frac{\Sigma u_\mu}{u_0}\right)^4 + \frac{315}{128}\left(\frac{\Sigma s_\mu}{s_0}\right)^4 + \frac{15}{32}\left(\frac{\Sigma u_\mu}{u_0}\right)^3 \frac{\Sigma s_\mu}{s_0} + \frac{35}{32}\left(\frac{\Sigma s_\mu}{s_0}\right)^3 \frac{\Sigma u_\mu}{u_0} + \frac{45}{64}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 \left(\frac{\Sigma s_\mu}{s_0}\right)^2 \right.$$
$$\left. - \left(\frac{\Sigma z_\mu}{\rho_0}\right) \left[\frac{105}{16}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 + \frac{45}{16}\left(\frac{\Sigma s_\mu}{s_0}\right)^2 + \frac{45}{8} \frac{\Sigma u_\mu}{u_0} \cdot \frac{\Sigma s_\mu}{s_0}\right] + \frac{15}{8}\left(\frac{\Sigma z_\mu}{\rho_0}\right)^4 \right\}$$

and

$$\frac{\kappa}{\rho_0^2}\left[-\frac{\Sigma z_\mu}{\rho_0} \left\{ \frac{35}{16}\left(\frac{\Sigma u_\mu}{u_0}\right)^3 + \frac{35}{16}\left(\frac{\Sigma s_\mu}{s_0}\right)^3 + \frac{45}{16}\left(\frac{\Sigma u_\mu}{u_0}\right)^2 \frac{\Sigma s_\mu}{s_0} + \frac{45}{16}\left(\frac{\Sigma s_\mu}{s_0}\right)^2 \frac{\Sigma u_\mu}{u_0} \right\} + \frac{15}{4}\left(\frac{\Sigma z}{\rho_0}\right)^3 \left\{ \frac{\Sigma u_\mu}{u_0} + \frac{\Sigma s_\mu}{s_0} \right\} \right].$$

The meanings of the various symbols have been explained in Chap. I.

The parts arising from Ω, namely, $\frac{\partial \Omega}{\partial s} \zeta^{-1}, -\frac{1}{2}\frac{\partial \Omega}{\partial z}$, are treated in like manner.

In the parts of equations (17), (18),

$$\zeta^{-1}(D^2 + 2\,\mathrm{m}\,D)\,\Sigma\, u_\mu, \qquad -D^2 \Sigma z_\mu,$$

we substitute

$$u_\mu = u_2, \qquad z_\mu = z_2,$$

and proceed as follows: in those parts of the arguments of u_2, z_2 which contain c, g we must use the values of c, g to the second order (that is, we must retain the parts of these two quantities which depend on e^2, e'^2, k^2, a^2); when the operations D^2, D have been performed the portions of the fourth order must be retained. In all other operations the parts c_0, g_0 of c, g will be sufficient.

The general procedure in performing the calculations and the methods of testing the results are in other respects the same as those explained in § 66.

97. The series finally obtained before proceeding to the solution of the linear differential equations are not, in the case of equation (17), the actual right-hand members, but series $\mathfrak{A}$, where

$$A=\tfrac{3}{4}\frac{\kappa u_0\zeta^{-1}}{\rho_0^{\,3}}\mathfrak{A}.$$

The reasons for this have been stated in § 71 (the symbol $\mathfrak{A}$ is there denoted by A_1).

In consequence of this change the series s_2, u_2, s_3, s_4 (§§ 68–70) must be replaced by series $\mathfrak{s}_2$, $\mathfrak{u}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, where

$$\mathfrak{u}_2=\tfrac{3}{4}\frac{\kappa u_0\zeta^{-1}}{\rho_0^{\,3}}u_2,\quad \mathfrak{s}_p=\tfrac{3}{4}\frac{\kappa u_0\zeta^{-1}}{\rho_0^{\,3}}s_p \qquad (p=2,\ 3,\ 4).$$

Further, as we shall find $u_\lambda\zeta^{-1}/\mathrm{a}$ directly, instead of finding u_λ/u_0 first, we use series $\mathfrak{U}_1$, $\mathfrak{U}_2$, $\mathfrak{U}_3$, $\mathfrak{U}_4$ instead of the series U_1, U_2, U_3, U_4, where

$$\mathfrak{U}_p=\frac{u_0\zeta^{-1}}{\mathrm{a}}U_p \qquad (p=1,\ 2,\ 3,\ 4).$$

The values of these eight new series are given below.

i.	$\mathfrak{s}_2$.	$\mathfrak{u}_2$.	$\mathfrak{s}_3$.	$\mathfrak{s}_4$.
4	+·00000 023	−·00000 003	−·00000 002	+·00000 003
3	+·00002 287	−·00000 402	−·00000 210	+·00000 300
2	+·00217 032	−·00048 522	−·00020 726	+·00028 048
1	+·17116 620	−·06341 516	−·01843 390	+·02151 153
0	−·29502 375	+·87776 191	−·87862 801	−·81835 400
−1	−·00233 553	+·01093 391	−·00581 778	−·00739 195
−2	−·00001 951	+·00011 479	−·00004 631	−·00006 191
−3	−·00000 017	+·00000 113	−·00000 038	−·00000 053
−4		+·00000 001		
Sum	−·12401 934	+·82490 732	−·90313 576	−·80401 335

i.	$\mathfrak{U}_1$.	$\mathfrak{U}_2$.	$\mathfrak{U}_3$.	$\mathfrak{U}_4$.
4		− ·00000 004		
3	−·00000 022	− ·00000 608	+ ·00000 021	−·00000 032
2	−·00002 946	− ·00089 987	+ ·00002 939	−·00004 410
1	−·00428 758	− ·16212 885	+ ·00454 712	−·00648 148
0	−·73409 342	+2·18355 94	+1	−·93146 358
−1	+·43443 130	+ ·00036 969	+ ·00869 575	+·03592 927
−2	+·00015 197	− ·00000 196	− ·00000 049	−·00000 147
−3	−·00000 036	− ·00000 001	− ·00000 001	−·00000 003
−4				
Sum	−·30382 777	+2·02089 23	+1·01327 197	−·90206 171

Section (ii). *Values of* $\mathfrak{A}$, $u_\lambda \zeta^{-1}/a\lambda$; A, $\iota z_\lambda/a\lambda$.

98. The following tables show the characteristics, arguments, and types of coefficients of the terms of the fourth order according to the scheme adopted in Section (iv), Chap. I. The numerical results are given below in the same order, and will be found in the §§ given in the first columns.

§	λ	Arguments.	Types of Coefficients in $u_\lambda\zeta^{-1}/a\lambda$.
99	e^4	$2i\pm 4c$; $2i\pm 2c$; $2i$	(ϵ^4), (ϵ'^4) ; $(\epsilon^3\epsilon')$, $(\epsilon\epsilon'^3)$; $(\epsilon^2\epsilon'^2)$
100	e^3e'	$2i\pm(3c+m)$; $2i\pm(3c-m)$; $2i\pm(c+m)$; $2i\pm(c-m)$	$(\epsilon^3\eta)$, $(\epsilon'^3\eta')$; $(\epsilon^3\eta')$, $(\epsilon'^3\eta)$; $(\epsilon^2\epsilon'\eta)$, $(\epsilon\epsilon'^2\eta')$; $(\epsilon^2\epsilon'\eta')$, $(\epsilon\epsilon'^2\eta)$
101	$e^2e'^2$	$2i\pm(2c+2m)$; $2i\pm(2c-2m)$; $2i\pm 2c$; $2i\pm 2m$; $2i$	$(\epsilon^2\eta^2)$, $(\epsilon'^2\eta'^2)$; $(\epsilon^2\eta'^2)$, $(\epsilon'^2\eta^2)$; $(\epsilon^2\eta\eta')$, $(\epsilon'^2\eta\eta')$; $(\epsilon\epsilon'\eta^2)$, $(\epsilon\epsilon'\eta'^2)$; $(\epsilon\epsilon'\eta\eta')$
102	ee'^3	$2i\pm(c+3m)$; $2i\pm(c-3m)$; $2i\pm(c+m)$; $2i\pm(c-m)$	$(\epsilon\eta^3)$, $(\epsilon'\eta'^3)$; $(\epsilon\eta'^3)$, $(\epsilon'\eta^3)$; $(\epsilon\eta^2\eta')$, $\epsilon'\eta\eta'^2)$; $(\epsilon\eta\eta'^2)$, $(\epsilon'\eta^2\eta')$
103	e'^4	$2i\pm 4m$; $2i+2m$; $2i$	(η^4), (η'^4) ; $(\eta^3\eta')$, $(\eta\eta'^3)$; $(\eta^2\eta'^2)$
104	e^2k^2	$2i\pm(2c+2g)$; $2i\pm(2c-2g)$; $2i\pm 2c$; $2i\pm 2g$; $2i$	(ϵ^2k^2), $(\epsilon'^2k'^2)$; $(\epsilon^2k'^2)$, (ϵ'^2k^2) ; (ϵ^2kk'), (ϵ'^2kk') ; $(\epsilon\epsilon'k^2)$, $(\epsilon\epsilon'k'^2)$; $(\epsilon\epsilon'kk')$
105	$ee'k^2$	$2i\pm(c+m+2g)$; $2i\pm(c+m-2g)$; $2i\pm(c-m+2g)$; $2i\pm(c-m-2g)$; $2i\pm(c+m)$; $2i\pm(c-m)$	$(\epsilon\eta k^2)$, $(\epsilon'\eta'k'^2)$; $(\epsilon\eta k'^2)$, $(\epsilon'\eta'k^2)$; $(\epsilon\eta'k^2)$, $(\epsilon'\eta k'^2)$; $(\epsilon\eta'k'^2)$, $(\epsilon'\eta k^2)$; $(\epsilon\eta kk')$, $(\epsilon'\eta'kk')$; $(\epsilon\eta'kk')$, $(\epsilon'\eta kk')$
106	e'^2k^2	$2i\pm(2m+2g)$; $2i\pm(2m-2g)$; $2i\pm 2m$; $2i\pm 2g$; $2i$	(η^2k^2), $(\eta'^2k'^2)$; $(\eta^2k'^2)$, (η'^2k^2) ; (η^2kk'), (η'^2kk') ; $(\eta\eta'k^2)$, $(\eta\eta'k'^2)$; $(\eta\eta'kk')$
107	k^4	$2i\pm 4g$; $2i\pm 2g$; $2i$	(k^4), (k'^4) ; (k^3k'), (kk'^3) ; $(k^2k'^2)$
108	$e^3\alpha$	$2i_1\pm 3c$; $2i_1\pm c$	$(\epsilon^3\alpha)$, $(\epsilon'^3\alpha)$; $(\epsilon^2\epsilon'\alpha)$, $(\epsilon\epsilon'^2\alpha)$
109	$e^2e'\alpha$	$2i_1\pm(2c+m)$; $2i_1\pm(2c-m)$; $2i_1\pm m$	$(\epsilon^2\eta\alpha)$, $(\epsilon'^2\eta'\alpha)$; $(\epsilon^2\eta'\alpha)$, $(\epsilon'^2\eta\alpha)$; $(\epsilon\epsilon'\eta\alpha)$, $(\epsilon\epsilon'\eta'\alpha)$
110	$ee'^2\alpha$	$2i_1\pm(c+2m)$; $2i_1\pm(c-2m)$; $2i_1\pm c$	$(\epsilon\eta^2\alpha)$, $(\epsilon'\eta'^2\alpha)$; $(\epsilon\eta'^2\alpha)$, $(\epsilon'\eta^2\alpha)$; $(\epsilon\eta\eta'\alpha)$, $(\epsilon'\eta\eta'\alpha)$
111	$e'^3\alpha$	$2i_1\pm 3m$; $2i_1\pm m$	$(\eta^3\alpha)$, $(\eta'^3\alpha)$; $(\eta^2\eta'\alpha)$, $(\eta\eta'^2\alpha)$
112	$ek^2\alpha$	$2i_1\pm(c+2g)$; $2i_1\pm(c-2g)$; $2i_1\pm c$	$(\epsilon k^2\alpha)$, $(\epsilon'k'^2\alpha)$; $(\epsilon k'^2\alpha)$, $(\epsilon'k^2\alpha)$; $(\epsilon kk'\alpha)$, $(\epsilon'kk'\alpha)$
113	$e'k^2\alpha$	$2i_1\pm(m+2g)$; $2i_1\pm(m-2g)$; $2i_1\pm m$	$(\eta k^2\alpha)$, $(\eta'k'^2\alpha)$; $(\eta k'^2\alpha)$, $(\eta'k^2\alpha)$; $(\eta kk'\alpha)$, $(\eta'kk'\alpha)$
114	$e^2\alpha^2$	$2i\pm 2c$; $2i$	$(\epsilon^2\alpha^2)$, $(\epsilon'^2\alpha^2)$; $(\epsilon\epsilon'\alpha^2)$
115	$ee'\alpha^2$	$2i\pm(c+m)$; $2i\pm(c-m)$	$(\epsilon\eta\alpha^2)$, $(\epsilon'\eta'\alpha^2)$; $(\epsilon\eta'\alpha^2)$, $(\epsilon'\eta\alpha^2)$
116	$k^2\alpha^2$	$2i\pm 2g$; $2i$	$(k^2\alpha^2)$, $(k'^2\alpha^2)$; $(kk'\alpha^2)$

§	λ	Arguments.	Types of Coefficients in $\iota z_\lambda/a\lambda$.
117	ke^3	$2i+g\pm 3c$; $2i+g\pm c$	$(k\epsilon^3)$, $(k\epsilon'^3)$; $(k\epsilon^2\epsilon')$, $(k\epsilon\epsilon'^2)$
118	ke^2e'	$2i+g\pm(2c+m)$; $2i+g\pm(2c-m)$; $2i+g\pm m$	$(k\epsilon^2\eta)$, $(k\epsilon'^2\eta')$; $(k\epsilon^2\eta')$, $(k\epsilon'^2\eta)$; $(k\epsilon\epsilon'\eta)$, $(k\epsilon\epsilon'\eta')$
119	kee'^2	$2i+g\pm(c+2m)$; $2i+g\pm(c-2m)$; $2i+g\pm c$	$(k\epsilon\eta^2)$, $(k\epsilon'\eta'^2)$; $(k\epsilon\eta'^2)$, $(k\epsilon'\eta^2)$; $(k\epsilon\eta\eta')$, $(k\epsilon'\eta\eta')$

§	λ	Arguments.	Types of Coefficients in $\iota z_\lambda/a\lambda$.
120	ke'^3	$2i+g\pm 3m\,;\ 2i+g\pm m$	$(k\eta^3), (k\eta'^3)\,;\ (k\eta^2\eta'), (k\eta\eta'^2)$
121	k^3e	$2i+3g\pm c\,;\ 2i+g\pm c$	$(k^3\epsilon), (k^3\epsilon')\,;\ (k^2k'\epsilon), (k^2k'\epsilon')$
122	k^3e'	$2i+3g\pm m\,;\ 2i+g\pm m$	$(k^3\eta), (k^3\eta')\,;\ (k^2k'\eta), (k^2k'\eta')$
123	$ke^2\alpha$	$2i_1+g\pm 2c\,;\ 2i_1+g$	$(k\epsilon^2\alpha), (k\epsilon'^2\alpha)\,;\ (k\epsilon\epsilon'\alpha)$
124	$kee'\alpha$	$2i_1+g\pm(c+m)\,;\ 2i_1+g\pm(c-m)$	$(k\epsilon\eta\alpha), (k\epsilon'\eta'\alpha)\,;\ (k\epsilon\eta'\alpha), (k\epsilon'\eta\alpha)$
125	$ke'^2\alpha$	$2i_1+3g\pm 2m\,;\ 2i_1+g$	$(k\eta^2\alpha), (k\eta'^2\alpha)\,;\ (k\eta\eta'\alpha)$
126	$k^3\alpha$	$2i_1+3g\,;\ 2i_1+g$	$(k^3\alpha)\,;\ (k^2k'\alpha)$
127	$ke\alpha^2$	$2i+g\pm c$	$(k\epsilon\alpha^2), (k\epsilon'\alpha^2)$
128	$ke'\alpha^2$	$2i+g\pm m$	$(k\eta\alpha^2), (k\eta'\alpha^2)$

The coefficients of $\iota z_\lambda/a\lambda$ change their signs when the corresponding arguments change their signs.

The following long-period terms have been obtained with the required accuracy by the method of Section (iii), Chap. V. :—

Arguments.	Coefficients.
$\pm(-2+2c)$,	$(\epsilon^3\epsilon')_{-1}$, $(\epsilon\epsilon'^3)_1$
$\pm(-2+2c-2m)$,	$(\epsilon^2\eta'^2)_{-1}$, $(\epsilon'^2\eta^2)_1$
$\pm(2c-2g)$,	$(\epsilon^2k'^2)_0$, $(\epsilon'^2k^2)_0$
$\pm(2+2m-2g)$,	$(\eta^2k'^2)_1$, $(\eta'^2k^2)_{-1}$

The values of V_λ for these terms, obtained by the method of Section (ii), Chap. V., agreed, as far as they went, with the values obtained by Section (iii); this agreement furnished a valuable test.

In the solution of the linear differential equations those sets of terms with arguments $2i$ have no small divisors; the continued approximation method was, therefore, employed for such terms instead of the method of Section (ii), Chap. V.* The continued approximation method was also employed in the solution of the differential equations for the terms with characteristics $e'^3\alpha$, $e^2\alpha^2$, $ee'\alpha^2$, $k^2\alpha^2$, and for all the terms in z.

The numerical results now follow. The values of $\mathfrak{U}_\lambda$ for the terms arising in u are given in two parts—those arising from the expansion of $\kappa u\zeta^{-1}/r^3$, denoted by $\mathfrak{K}_\lambda$, and those arising from $-\left\{(D^2+2mD)\Sigma u_\mu+\dfrac{\partial\Omega_1}{\partial s}\right\}$, denoted by $\mathfrak{L}_\lambda$. Then

$$\mathfrak{U}_\lambda=\mathfrak{K}_\lambda+\mathfrak{L}_\lambda=\frac{4\rho_0^3}{3\kappa u_0\zeta^{-1}}A_\lambda \qquad \mathfrak{K}_\lambda=\frac{4\rho_0^3}{3\kappa u_0\zeta^{-1}}K^\lambda,$$

$$\mathfrak{L}_\lambda=\frac{4\rho_0^3}{3\kappa u_0\zeta^{-1}}\times\text{part, characteristic } \lambda, \text{ in} -\left\{(D^2+2mD)\Sigma u_\mu+\frac{\partial\Omega_1}{\partial s}\right\}\zeta^{-1},$$

where A_λ is the right-hand member of equation (17), Chap. I.

* The method for obtaining the results given in the section referred to has been published in the *Camb. Phil. Trans.* vol. xviii. pp. 94–106, under the title, "On the Solution of a Pair of Simultaneous Linear Differential Equations, which occur in the Lunar Theory."

In the case of z, we have

$$A_\lambda = K_\lambda + L_\lambda,$$

where A_λ is the right-hand member of equation (18), Chap. I., K_λ is the part arising from the expansion of $\varkappa z/r^3$, and L_λ is the part arising from the expansion of $-D^2\Sigma z_\mu - \frac{1}{2}\frac{\partial\Omega}{\partial z}$.

The numbers are the coefficients corresponding to the power of ζ (that is, the argument) which is placed at the head of each column or pair of columns.

The suffix of i_1 is omitted in the tables. Further details concerning the results will be found in Chaps. I.–IV.

99. *Characteristic* e^4.

Values of $\mathfrak{K}+\mathfrak{L}$. For arguments $2i\pm 4c$, $2i$, $\mathfrak{L}=0$.

i.	$2i+4c$.	$2i-4c$.	$2i$.
5		+·00003 95	+·00000 28
4	+·00000 02	+·00103 13	+·00013 62
3	+·00001 08	+·00204 86	+·00587 17
2	+·00059 44	+·00427 24	+·14884 07
1	+·02594 23	−·03457 94	−·06923 47
0	+·67584 28	+·05789 84	−·06099 65
−1	−·22643 43	+·00198 63	−·00928 89
−2	+·00669 89	+·00004 43	+·01353 65
−3	+·00045 68	+·00000 07	+·00045 91
−4	+·00010 24		+·00000 99
−5	+·00000 34		+·00000 01
Sum	+·48321 77	+·03274 21	+·02933 69

	$2i+2c$.		$2i-2c$.	
5	+·00000 01		+·00001 79	
4	+·00000 86		+·00077 93	
3	+·00046 28		+·01982 81	+·00000 10
2	+·02001 44	−·00000 02	+·00431 95	+·00015 18
1	+·51237 13	−·00002 62	+·00596 01	+·00018 42
0	−·24760 91	−·00371 87	−·03008 47	−·00041 35
−1	+·00727 51	+·00098 30	+·04523 32	−·00000 16
−2	+·00121 42	+·00001 06	+·00154 64	
−3	+·00187 28		+·00003 45	
−4	+·00006 20		+·00000 04	
−5	+·00000 13			
Sum	+·29567 35	−·00275 15	+·04763 47	−·00007 81

Characteristic e^4.

Values of

i.	$(\epsilon^4)_i$.	$(\epsilon'^4)_i$.	$(\epsilon^3\epsilon')_i$.	$(\epsilon\epsilon'^3)_i$.	$(\epsilon^2\epsilon'^2)_i$.
4		+·00003 9		+·00001 9	+·00000 24
3		+·00020 3	+·00000 8	+·00069 4	+·00013 21
2	+·00001 1	+·00014 3	+·00043 8	+·00034 4	+·00489 88
1	+·00054 4	−·00077 4	+·01597 2	−·00172 8	−·00639 9
0	+·02003 5	+·00145 3	−·02093 6	+·00573 7	−·01524 5
−1	−·01647 0	+·00002 5	+·00474 6	+·00118 3	+·00222 8
−2	+·00217 0		+·00039 2	+·00001 9	+·00036 48
−3	+·00004 0		+·00005 1		+·00000 57
−4	+·00000 3		+·00000 2		+·00000 01
Sum	+·00633 3	+·00108 9	+·00067 3	+·00626 8	−·01401 2

100. *Characteristic* e^3e'.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+3c+m$.		$2i-3c-m$.	
5			+·00007 26	
4	− ·00000 05		+·00276 44	
3	− ·00002 51		+·05506 64	−·00000 21
2	− ·00128 53		+·05714 34	−·00000 99
1	− ·04820 09	−·00000 16	−·12700 10	+·00002 49
0	− ·91127 06	−·00023 27	−·10600 69	−·00020 08
−1	− ·44433 98	−·00003 60	−·00429 96	+·00023 15
−2	+ ·02661 37	+·00007 60	−·00010 63	+·00000 40
−3	+ ·00794 42	−·00006 92	−·00000 22	+·00000 01
−4	+ ·00027 94	−·00001 50		
−5	+ ·00000 63	−·00000 02		
Sum	−1·37027 86	−·00027 87	−·12236 92	+·00004 77

Characteristic e^3e'.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+3c-m$.		$2i-3c+m$.	
5			− ·00001 05	
4	+ ·00000 25		− ·00040 31	
3	+ ·00013 46		− ·00782 47	− ·00000 20
2	+ ·00624 15		+ ·01156 09	− ·00001 09
1	+ ·18642 58	− ·00000 16	− ·06276 16	− ·00007 40
0	+ 1·09809 98	− ·00023 36	+ ·13395 42	+ ·00117 18
− 1	− ·19917 10	+ ·00019 86	+ ·01566 24	− ·00162 22
− 2	+ ·00505 43	− ·00002 30	+ ·00049 22	− ·00002 77
− 3	− ·00101 90	+ ·00000 96	+ ·00001 02	− ·00000 04
− 4	− ·00003 82	+ ·00000 22	+ ·00000 02	
− 5	− ·00000 02			
Sum	+ 1·09573 01	− ·00004 78	+ ·09068 02	− ·00056 54

i.	$2i+c+m$.		$2i-c-m$.	
5	− ·00000 03		+ ·00001 97	
4	− ·00002 53		+ ·00096 49	
3	− ·00077 45		+ ·03453 15	+ ·00000 01
2	− ·02927 29	− ·00000 08	+ ·57798 75	+ ·00000 04
1	− ·57119 50	− ·00012 14	+ ·09193 66	+ ·00193 32
0	− ·63687 77	+ ·00135 94	− ·20284 99	+ ·00026 58
− 1	+ ·03451 56	+ ·00209 42	− ·06986 67	− ·00025 82
− 2	+ ·07900 65	+ ·00021 27	− ·00267 69	+ ·00014 10
− 3	+ ·00332 03	− ·00020 68	− ·00006 43	+ ·00000 24
− 4	+ ·00008 35	− ·00000 34	− ·00000 09	
− 5	+ ·00000 16	− ·00000 01		
Sum	− 1·12120 82	+ ·00333 38	+ ·42998 15	+ ·00208 47

Characteristic e^3e'.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+c-m$.		$2i-c+m$.	
5	+ ·00000 17		−·00000 33	
4	+ ·00008 90		−·00015 73	
3	+ ·00426 64		−·00593 15	−·00000 02
2	+ ·14161 61	−·00000 22	−·11616 23	−·00003 40
1	+1·78220 15	−·00030 33	−·03064 08	−·00029 97
0	+ ·20914 40	−·00135 79	+·07494 72	−·00056 43
−1	− ·01155 70	−·00034 43	+·23267 55	+·00165 22
−2	− ·01469 83	−·00003 47	+·01289 11	−·00098 79
−3	− ·00055 16	+·00002 95	+·00035 40	−·00001 68
−4	− ·00001 36	+·00000 05	+·00000 77	−·00000 02
−5	− ·00000 04		+·00000 02	
Sum	+2·11049 78	−·00201 24	+·16798 05	−·00025 09

Values of

i.	$(\epsilon^3\eta)_i$.	$(\epsilon'^3\eta')_i$.	$(\epsilon^3\eta')_i$.	$(\epsilon'^3\eta)_i$.
4		+·00008 7		−·00001 3
3		+·00321 8	+·00000 2	−·00040 9
2	−·00002 4	+·00936 8	+·00011 0	+·00307 6
1	−·00125 5	−·00590 5	+·00445 3	−·00913 3
0	−·04042 4	−·00396 9	+·05193 1	+·00576 7
−1	−·06117 4	−·00006 1	−·02861 3	+·00024 4
−2	+·00900 2	−·00000 1	−·00166 0	+·00000 5
−3	+·00036 0		−·00004 7	
−4	+·00000 4			
Sum	−·09351 1	+·00273 7	+·02617 6	−·00046 3

Characteristic e^3e'.

Values of

i.	$(\epsilon^2\epsilon'\eta)_i$.	$(\epsilon\epsilon'^2\eta')_i$.	$(\epsilon^2\epsilon'\eta')_i$.	$(\epsilon\epsilon'^2\eta)_i$.
4		+·00002 0	+·00000 1	−·00000 2
3	−·00001 5	+·00101 2	+·00008 3	−·00017 0
2	−·00080 0	+·03124 7	+·00380 0	−·00593 2
1	−·02711 8	+·00408 0	+·09003 4	−·01970 4
0	−·08633 0	−·04425 8	+·01113 1	+·07463 4
−1	+·03812 8	−·00268 8	+·03997 4	+·01114 3
−2	+·00390 1	−·00003 8	−·00060 0	+·00019 9
−3	+·00004 9	−·00000 1	−·00000 8	+·00000 4
−4	+·00000 1		−·00000 1	
Sum	−·07218 4	−·01062 6	+·14441 4	+·06017 2

101. *Characteristic* $e^2e'^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2c+2m$.		$2i-2c-2m$.	
5			+·00004 49	
4			+·00204 02	
3	+·00001 55		+·06459 05	−·00000 10
2	+·00057 27		+·86143 06	−·00016 58
1	+·00793 85	+·00000 40	−·47189 56	−·00028 97
0	−·37149 07	−·00042 61	−·07081 97	−·00100 51
−1	−·66500 19	−·00105 94	+·00030 30	−·00034 89
−2	+·20280 05	−·00298 76	+·00004 17	−·00000 73
−3	+·00849 83	−·00133 65	+·00000 11	−·00000 01
−4	+·00021 90	−·00002 05		
−5	+·00000 49	−·00000 03		
Sum	−·81644 32	−·00582 64	+·38573 67	−·00181 79

Characteristic $e^2e'^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i	$2i+2c-2m.$		$2i-2c+2m.$	
5			−·00000 05	
4	+ ·00001 24		−·00001 73	
3	+ ·00072 90		−·00278 01	+·00000 02
2	+ ·02854 11	−·00000 02	−·18735 97	−·00001 29
1	+ ·59904 28	−·00002 69	+·14699 65	−·00046 45
0	+1·26100 51	−·00126 11	+·25718 37	+·00742 25
−1	+ ·13905 30	+·00089 50	+·06791 35	−·01218 97
−2	− ·03928 35	+·00015 55	+·00269 03	−·00031 19
−3	− ·00052 55	−·00001 86	+·00006 59	−·00000 46
−4	− ·00000 42	−·00000 03	+·00000 07	−·00000 01
−5	− ·00000 01			
Sum	+1·98857 01	−·00025 66	+·28469 30	−·00556 10

i	$2i+2c.$		$2i-2c.$	
5			−·00001 20	
4	−·00000 48		−·00058 51	−·00000 01
3	−·00024 76		−·01832 28	−·00001 42
2	−·01039 93	+·00000 27	−·23064 49	−·00208 94
1	−·25413 96	+·00031 50	−·02504 78	−·00311 93
0	−·54218 73	+·04731 79	−·09083 51	+·00876 11
−1	−·02036 16	−·01141 91	−·02701 79	+·00555 46
−2	−·04993 83	−·00067 12	−·00096 10	+·00010 91
−3	−·00225 16	+·00038 18	−·00002 02	+·00000 15
−4	−·00006 08	+·00000 59	−·00000 05	+·00000 01
−5	−·00000 07	+·00000 01		
Sum	−·87959 16	+·03593 31	−·39344 73	+·00920 34

Characteristic $e^2e'^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2m$.		$2i-2m$.	
5			+ ·00000 79	
4	+ ·00000 27		+ ·00036 03	
3	+ ·00002 85		+ ·01554 27	− ·00000 01
2	− ·01244 90	+ ·00000 16	+ ·41891 21	− ·00001 06
1	− 1·03332 56	− ·00017 67	+ 3·40300 52	− ·00103 89
0	− ·11065 41	+ ·00096 19	− ·10257 34	+ ·00056 05
−1	+ ·75079 48	+ ·00658 16	− ·20699 71	− ·00117 71
−2	+ ·05144 80	− ·00865 61	− ·00256 42	− ·00016 26
−3	+ ·00157 54	− ·00016 17	− ·00001 78	− ·00000 31
−4	+ ·00003 37	− ·00000 22	+ ·00000 03	
−5	+ ·00000 08			
Sum	− ·35254 48	− ·00146 16	+ 3·52567 60	− ·00183 19

i.	$2i$.	
5	− ·00000 19	
4	− ·00011 21	
3	− ·00497 96	− ·00000 01
2	− ·14155 47	− ·00000 92
1	− 1·23587 32	− ·00076 02
0	− ·39760 68	− ·00044 40
−1	− ·24832 43	+ ·00439 86
−2	− ·01642 02	+ ·00294 30
−3	− ·00048 60	+ ·00005 04
−4	− ·00001 16	+ ·00000 07
−5	− ·00000 03	
Sum	− 2·04537 07	+ ·00617 92

Characteristic $e^2e'^2$.

Values of

i.	$(\epsilon^2\eta^2)_i$.	$(\epsilon'^2\eta'^2)_i$.	$(\epsilon^2\eta'^2)_i$.	$(\epsilon'^2\eta^2)_i$.
4		+·00005 3		−·00000 1
3	+·00000 2	+·00275 9	+·00001 2	−·00016 1
2	+·00001 0	+·08685 8	+·00062 3	−·01601 0
1	+·00011 0	+·07421 6	+·02043 7	−·14196
0	−·02678 9	−·00684 1	+·10603 8	+·03200 2
−1	−·28583 1	−·00000 3	+·21767	+·00135 5
−2	+·02348 4	+·00000 1	−·00396 8	+·00003 4
−3	+·00009 9		−·00001 0	+·00000 1
−4	+·00000 4			
Sum	−·28891 1	+·15704 3	+·34080	−·12474

i.	$(\epsilon^2\eta\eta')_i$.	$(\epsilon'^2\eta\eta')$.	$(\epsilon\epsilon'\eta^2)_i$.	$(\epsilon\epsilon'\eta'^2)_i$.	$(\epsilon\epsilon'\eta\eta')_i$.
4		−·00001 5		+·00000 7	−·00000 20
3	−·00000 4	−·00072 8	−·00000 3	+·00037 8	−·00011 78
2	−·00022 4	−·02068 2	−·00073 6	+·01603 8	−·00514 67
1	−·00815 4	+·05116	−·07832 4	+·30827 9	−·10410 4
0	−·03958 2	−·00131 0	−·18186	+·16309	−·09951 9
−1	−·05663	−·00042 6	+·10559 3	−·02533 0	−·02118 2
−2	−·00702 8	−·00001 1	+·00083 1	−·00004 7	−·00021 77
−3	−·00002 2		+·00002 1		−·00000 58
−4	−·00000 2				−·00000 01
Sum	−·11165	+·02799	−·15448	+·46242	−·23029 5

102. *Characteristic* ee'^3.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+c+3m$.		$2i-c-3m$.	
5			+·00001 11	
4			+·00057 24	
3	−·00000 23		+·02308 42	−·00000 02
2	+·00001 06		+·52094 60	−·00002 74
1	+·00353 66	+·00002 17	+·26017 23	−·00414 10
0	−·04803 63	−·00372 49	−·01662 47	+·00308 04
−1	+·13629 36	+·13899 01	+·00047 49	−·00034 45
−2	+·09169 96	−·04090 63	+·00000 48	+·00000 03
−3	+·00306 65	−·00057 78	−·00000 01	
−4	+·00006 57	−·00000 71		
−5	+·00000 11	−·00000 01		
Sum	+·18663 51	+·09379 56	+·78864 09	−·00143 24

i.	$2i+c-3m$.		$2i-c+3m$.	
5				
4	+ ·00004 19		+·00000 09	
3	+ ·00205 95		+·00012 66	
2	+ ·07211 99	−·00000 12	+·00832 00	+·00000 13
1	+1·18317 59	−·00015 04	+·10347 72	−·00023 73
0	+ ·27486 46	−·00493 83	+·11154 53	+·02718 31
−1	+ ·03165 59	−·00125 28	+·20318 44	−·06705 07
−2	+ ·00139 35	−·00057 45	+·00898 11	−·00176 45
−3	+ ·00002 18	−·00000 18	+·00022 34	−·00002 60
−4	+ ·00000 02		+·00000 45	−·00000 03
−5				
Sum	+1·56533 32	−·00691 90	+·43586 34	−·04189 44

Characteristic ee'^3.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	2*i*+c+m.		2*i*−c−m.	
5			−·00000 44	
4	+·00000 13		−·00024 16	
3	+·00010 29		−·00971 52	−·00000 34
2	+·00302 38	−·00000 20	−·21574 67	−·00041 10
1	+·01409 57	−·00027 39	−·37164 88	−·02742 72
0	−·15495 99	−·01772 85	−·04966 63	+·02712 44
−1	−·15945 22	−·03853 74	+·00145 78	−·00097 05
−2	−·03562 96	+·01719 29	+·00029 53	−·00008 42
−3	−·00122 46	+·00024 26	+·00000 94	−·00000 14
−4	−·00002 66	+·00000 30		
−5	−·00000 04	+·00000 01		
Sum	−·33406 96	−·03910 32	−·64526 05	−·00177 33

i.	2*i*+c−m.		2*i*−c+m.	
5				
4	−·00001 78		+·00001 34	
3	−·00094 85	+·00000 01	−·00001 01	+·00000 05
2	−·03389 63	+·00001 58	−·03797 42	+·00006 02
1	−·56836 53	+·00196 08	−·14814 17	+·00318 13
0	−·19411 40	+·01863 26	−·06658 97	+·02765 65
−1	−·05160 44	+·00491 80	−·09074 64	+·03547 34
−2	−·00689 49	+·00272 61	−·00401 73	+·00083 57
−3	−·00005 02	−·00000 95	−·00009 86	+·00001 19
−4	+·00000 01	−·00000 03	−·00000 13	+·00000 01
−5				
Sum	−·85589 13	+·02824 36	−·34756 59	+·06721 96

Characteristic ee'^3.

Values of

i.	$(\epsilon\eta^3)_i$.	$(\epsilon'\eta'^3)_i$.	$(\epsilon\eta'^3)_i$.	$(\epsilon'\eta^3)_i$.
4		+·00001 4	+·00000 1	
3		+·00078 6	+·00004 2	+·00000 4
2	−·00000 1	+·03522 1	+·00212 8	+·00037 4
1	−·00007 8	+·78390	+·06613 8	+·00357
0	−·04064	+·11111	+·17385	−·28327
−1	−1·52648	−·00001 2	+·03836	+·00097 5
−2	−·00492 4		−·00003 4	+·00014 7
−3	+·00004 7			+·00000 3
−4	+·00000 1			
Sum	−1·57208	+·93102	+·28049	−·27820

i.	$(\epsilon\eta^2\eta')_i$.	$(\epsilon'\eta\eta'^2)_i$.	$(\epsilon\eta\eta'^2)_i$.	$(\epsilon'\eta^2\eta')_i$.
4		−·00000 6		
3	+·00000 1	−·00031 1	−·00001 9	−·00000 5
2	+·00006 4	−·01336 8	−·00094 1	−·00116 8
1	−·00045 1	−·28623	−·02764 6	+·12094
0	−·19733	+·54293	+·23524	−·74298
−1	+·50408	−·00002 3	−·41624	+·00049 2
−2	+·00184 5	+·00000 3	−·00008 8	−·00005 7
−3	−·00001 6		−·00000 1	−·00000 1
−4	−·00000 1			
Sum	+·30819	+·24300	−·20970	−·62278

103. *Characteristic e'^4.*

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+4$m.		$2i-4$m.	
5			+·00000 13	
4			+ 00005 52	
3			+·00257 71	
2	+·00001 02	−·00000 01	+·07920 62	−·00000 22
1	+·00132 73	+·00014 65	+·13198 73	−·00036 11
0	+·03140 86	−·00886 25	+·02357 85	−·02153 32
−1	+·02695 70	−·39736 75	+·00022 85	−·00052 41
−2	+·01319 27	−·00549 20	+·00000 28	+·00000 07
−3	+·00035 92	−·00006 61		
−4	+·00000 75	−·00000 08		
−5	+·00000 01			
Sum	+·07326 26	−·41164 25	+·23763 69	−·02241 99

i.	$2i+2$m.		$2i-2$m.	
5				
4			−·00003 04	
3	−·00000 81		−·00143 50	
2	−·00003 41	+·00000 03	−·04376 96	−·00000 10
1	−·00080 80	+·00000 32	−·15494 59	−·00034 21
0	−·02145 30	−·00619 14	−·01687 08	+·00132 00
−1	−·03026 82	+·24551 88	−·00020 08	+·00031 30
−2	−·00693 33	+·00294 84	−·00000 49	−·00000 01
−3	−·00019 14	+·00003 64	−·00000 11	+·00000 02
−4	−·00000 35	+·00000 04		
−5				
Sum	−·05969 96	+·24231 61	−·21725 85	+·00129 00

Characteristic e'^4.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i$.	
5		
4	+·00000 53	
3	+·00021 42	
2	+·00536 69	+·00000 10
1	+·01053 21	−·00004 58
0	−·01872 34	−·00267 32
−1	+·00169 56	−·01129 18
−2	+·00078 57	−·00037 20
−3	+·00002 50	−·00000 57
−4	+·00000 04	−·00000 01
−5		
Sum	−·00009 82	−·01438 76

Values of

i.	$(\eta^4)_i$.	$(\eta'^4)_i$.	$(\eta^3\eta')_i$.	$(\eta\eta'^3)_i$.	$(\eta^2\eta'^2)_i$.
3		+·00007 5		−·00003 8	+·00000 50
2		+·00398 6	+·00000 1	−·00198 9	+·00019 44
1	+·00001 0	+·14687 7	+·00018 1	−·06580 0	+·00245 9
0	−·02698	+·04747	+·05634	−·08045	−·00536 0
−1	−·61854 2	−·00019 2	+·26249 4	−·00019 8	−·00885 8
−2	+·00019 2	+·00000 1	−·00006 7		+·00000 37
−3	+·00000 6		−·00000 3		+·00000 03
Sum	−·64531	+·19822	+·31895	−·14848	−·01155 6

104. *Characteristic* e^2k^2.

Values of $\mathfrak{K}+\mathfrak{L}$. For arguments $2i\pm 2c\pm 2g$, $2i$, $\mathfrak{L}=0$.

i.	$2i+2c+2g$.	$2i-2c-2g$.	$2i+2c-2g$.	$2i-2c+2g$.
5		+ ·00000 50		−·00000 07
4	−·00000 01	+ ·00021 12	+·00000 11	+·00000 11
3	+·00000 02	+ ·00430 10	+·00002 24	+·00000 41
2	+·00007 85	+ ·02174 09	−·00175 00	−·00591 44
1	+·00500 53	− ·18229 57	−·27258 99	−·86503 82
0	+·22308 59	+1·17697 36	+·53053 65	+·55015 59
−1	−·69815 36	+ ·03394 87	−·15877 59	−·05478 89
−2	+·03225 26	+ ·00062 72	+·04235 33	+·00037 20
−3	+·00096 80	+ ·00000 89	+·00128 22	+·00002 35
−4	+·00006 01		+·00002 42	+·00000 10
−5	+·00000 17		+·00000 03	
Sum	−·43670 14	+1·05552 08	+·14110 42	−·37518 46

i.	$2i+2c$.		$2i-2c$.	
5			−·00000 14	
4	− ·00000 04		−·00010 32	+·00000 01
3	− ·00002 44		−·00525 50	+·00002 08
2	− ·00123 16	−·00000 45	−·16771 28	+·00304 27
1	− ·02814 68	−·00052 60	+·04896 69	+·00369 31
0	+1·68846 77	−·07456 96	+·68013 38	−·00829 12
−1	+ ·04541 74	+·01971 25	+·07005 51	−·00003 14
−2	− ·03106 66	+·00021 32	+·00187 82	−·00000 05
−3	+ ·00182 12	+·00000 05	+·00003 40	
−4	+ ·00006 20		+·00000 08	
−5	+ ·00000 19			
Sum	+1·67530 04	−·05517 39	+·62799 64	−·00156 64

Characteristic e²k².

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2g$.		$2i-2g$.		$2i$.
5	+ ·00000 01		+ ·00000 04		− ·00000 04
4	+ ·00000 09		+ ·00002 74		− ·00001 39
3	+ ·00001 91		+ ·00068 73		− ·00076 69
2	+ ·00099 15		− ·03325 97	+ ·00000 13	− ·03247 94
1	+ ·01302 76	− ·00000 01	+ ·28101 50	+ ·00076 36	− ·57621 43
0	− 4·51169 89	− ·00009 89	− ·67308 93	− ·01560 86	− 1·44031 63
−1	+ ·35823 71	+ ·00168 86	+ ·44117 02	− ·00003 52	− ·05952 65
−2	− ·00820 20	+ ·00000 54	+ ·01307 25	− ·00000 01	+ ·02225 23
−3	+ ·00035 11		+ ·00024 35		+ ·00067 31
−4	+ ·00001 18		+ ·00000 49		+ ·00001 20
−5	+ ·00000 02		+ ·00000 01		+ ·00000 01
Sum	− 4·14726 15	+ 00159 50	+ ·02987 23	− ·01487 90	− 2·08638 02

Values of

i.	$(\epsilon^2 k^2)_i$.	$(\epsilon'^2 k'^2)_i$.	$(\epsilon^2 k'^2)_i$.	$(\epsilon'^2 k^2)_i$.
4		+ ·00001 0		
3		+ ·00042 2		− ·00000 7
2		+ ·00109 5	− ·00017 0	− ·00079 0
1	+ ·00002 4	− ·03189 0	− ·02398 8	− ·07175 9
0	+ ·00085 7	+ ·09359 9	− ·07730	+ ·34043
−1	− ·04605 1	+ ·00112 0	− ·01021 8	− ·00454 5
−2	+ ·01017 3	+ ·00001 1	+ ·00422 4	+ ·00008 9
−3	+ ·00009 1		+ ·00004 8	+ ·00000 1
−4	+ ·00000 4		+ ·00000 1	
Sum	− ·03490 2	+ ·06436 7	− ·10740	+ ·26342

Characteristic e^2k^2.

Values of

i.	$(\epsilon^2 kk')_i$.	$(\epsilon'^2 kk')_i$.	$(\epsilon\epsilon' k^2)_i$.	$(\epsilon\epsilon' k'^2)_i$.	$(\epsilon\epsilon' kk')_i$.
4		−·00000 4		+·00000 1	−·00000 03
3	−·00000 1	−·00027 1		+·00000 7	−·00002 43
2	−·00003 2	−·01602 8	−·00003 2	−·00323 2	−·00151 42
1	−·00102 5	−·03306	−·00370 3	−·02056	−·05637 1
0	+·08873 1	+·25316 0	−·35724 2	+·01475 2	−·36003 0
−1	+·05520	+·00622 2	+·16072	+·03914 8	+·03525 8
−2	+·00101 8	+·00006 9	−·00146 8	+·00045 8	+·00247 04
−3	+·00026 3	+·00000 1	+·00003 8	+·00000 5	+·00002 72
−4	+·00000 2		+·00000 1		+·00000 03
Sum	+·14416	+·21009	−·20169	+·03058	−·38018 4

105. *Characteristic* $ee'k^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+c+m+2g$.		$2i-c-m-2g$.	
5			+·00000 42	
4	− ·00000 05		+·00021 14	
3	− ·00000 08		+·00877 60	
2	− ·00006 92		+·16983 13	−·00002 30
1	− ·00356 07	+·00000 01	−·34429 60	+·00016 72
0	− ·08618 70	+·00005 01	−·36413 27	−·00179 20
−1	−1·19953 59	−·00937 10	−·02535 01	+·00000 33
−2	+ ·07743 85	+·00064 02	−·00060 82	+·00000 01
−3	+ ·00306 69	−·00015 20	−·00001 04	
−4	+ ·00008 06	−·00000 16	−·00000 03	
−5	+ ·00000 20			
Sum	−1·20876 61	−·00883 42	−·55557 48	−·00164 44

Characteristic $\mathrm{ee'k^2}$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+\mathrm{c}+\mathrm{m}-2\mathrm{g}$.		$2i-\mathrm{c}-\mathrm{m}+2\mathrm{g}$.	
5			+·00000 08	
4	−·00000 21		+·00000 08	
3	−·00018 74		+·00008 57	
2	−·00927 28	−·00000 20	+·00432 54	
1	−·30774 52	+·00027 64	+·09701 39	−·00002 07
0	+·00314 35	−·01050 06	−·00505 45	+·00286 25
−1	+·70749 51	+·01781 40	−·10767 95	+·00265 37
−2	+·03875 60	+·00005 33	−·00459 77	−·00016 85
−3	+·00089 18	+·00000 02	−·00010 12	+·00000 02
−4	+·00001 51		−·00000 13	
−5	+·00000 03			
Sum	+·43309 43	+·00764 13	−·01600 76	+·00532 72

i.	$2i+\mathrm{c}-\mathrm{m}+2\mathrm{g}$.		$2i-\mathrm{c}+\mathrm{m}-2\mathrm{g}$.	
5			−·00000 04	
4	−·00000 05		−·00004 96	
3	+·00000 80		−·00252 47	
2	+·00045 81		−·06776 15	−·00002 05
1	+·02115 16		+·09205 61	−·00026 88
0	+·21254 26	−·00002 16	+·47352 52	+·00138 82
−1	+·30194 12	+·00179 35	+·13623 37	−·00006 50
−2	−·02729 45	−·00014 91	+·00367 59	−·00000 08
−3	−·00103 08	+·00002 15	+·00006 67	
−4	−·00002 32	+·00000 02	+·00000 03	
−5	−·00000 01			
Sum	+·50775 24	+·00164 45	+·63522 17	+·00103 31

Characteristic ee'/k^2.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+c-m-2g$.		$2i-c+m+2g$.	
5	+·00000 03			
4	+·00001 31		−·00000 08	
3	+·00054 61		−·00000 82	
2	+·01389 61	−·00000 20	−·00027 59	
1	+·47952 76	+·00019 55	+·00611 99	−·00000 46
0	−·03394 87	−·00262 24	−·17243 46	+·00074 49
−1	−·08746 22	−·00288 95	+·18225 78	−·02624 26
−2	−·00530 01	−·00000 88	+·00774 06	+·00118 16
−3	−·00012 49		+·00027 29	−·00000 14
−4	−·00000 19		+·00000 73	
−5				
Sum	+·36714 54	−·00532 72	+·02367 90	−·02432 21

i.	$2i+c+m$.		$2i-c-m$.	
5			−·00000 18	
4	+ ·00000 01		−·00006 09	
3	+ ·00003 75		−·00323 90	+·00000 50
2	+ ·00182 63	+·00000 37	−·10552 73	+·00060 49
1	+ ·04846 15	+·00044 97	−·59901 89	+·03886 04
0	+1·24970 06	+·02492 64	+·37321 43	+·00038 74
−1	− ·30203 78	+·00957 00	−·11431 89	−·00019 28
−2	+ ·06215 35	+·00498 51	−·00316 98	−·00001 52
−3	+ ·00254 52	+·00004 12	−·00005 58	−·00000 02
−4	+ ·00005 27	+·00000 04	−·00000 05	
−5	+ ·00000 06			
Sum	+1·06274 02	+·03997 65	−·45217 86	+·03964 95

Characteristic $ee'k^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+c-m$.		$2i-c+m$.	
5	+ ·00000 03		− ·00000 03	
4	− ·00000 41		+ ·00000 95	
3	− ·00021 94	− ·00000 02	+ ·00052 42	− ·00000 07
2	− ·01004 62	− ·00002 58	+ ·01909 27	− ·00008 59
1	− ·19834 06	− ·00318 89	+ ·45975 74	− ·00548 87
0	− 1·20621 23	− ·03223 59	− ·47842 15	− ·01404 66
− 1	+ ·16517 85	− ·00201 12	+ ·20668 08	+ ·00052 86
− 2	− ·02140 03	− ·00073 39	+ ·01059 82	+ ·00010 47
− 3	− ·00061 57	− ·00000 60	+ ·00024 04	+ ·00000 13
− 4	− ·00001 18		+ ·00000 40	
− 5	− ·00000 03			
Sum	− 1·27167 19	− ·03820 19	+ ·21848 54	− ·01898 73

Values of

i.	$(\epsilon\eta k^2)_i$.	$(\epsilon'\eta' k'^2)_i$.	$(\epsilon\eta k'^2)_i$.	$(\epsilon'\eta' k^2)_i$.
4		+ ·00000 7		
3		+ ·00047 5	− ·00000 5	− ·00000 1
2		+ ·01895	− ·00020 1	− ·00011 6
1	+ ·00000 2	− ·08133	+ ·01155	− ·01009 1
0	+ ·00079 3	− ·05729 2	− ·13724	+ ·04755
− 1	− ·14433	− ·00121 6	+ ·15104 1	− ·18609
− 2	+ ·04641	− ·00001 5	+ ·00216 2	− ·00070 8
− 3	+ ·00043 6		+ ·00002 2	− ·00000 4
− 4	+ ·00000 4			
Sum	− ·09669	− ·12042	+ ·02733	− ·14946

Characteristic $ee'\mathrm{k}^2$.

Values of

i.	$(\epsilon\eta' k^2)_i$.	$(\epsilon'\eta k'^2)_i$.	$(\epsilon\eta' k'^2)_i$.	$(\epsilon'\eta k^2)_i$.
4		−·00000 2	+·00000 1	
3		−·00015 1	+·00000 8	
2	+·00000 2	−·01142	−·00028 7	+·00002 0
1	+·00014 0	+·03952	−·12043	+·00146 6
0	+·00216 2	+·08429 0	+·14853	−·07174
−1	+·03509	+·00694 7	− 01646 7	+·52873
−2	−·00641	+·00009 2	−·00027 5	+·00182 4
−3	−·00013 7	+·00000 1	−·00000 3	+·00001 3
−4	−·00000 1			
Sum	+·03085	+·11928	+·01108	+·46031

i.	$(\epsilon\eta kk')_i$.	$(\epsilon'\eta' kk')_i$.	$(\epsilon\eta' kk')_i$.	$(\epsilon'\eta kk')_i$.
4		− ·00000 1		
3	+ ·00000 2	− ·00015 0	−·00000 6	+ ·00001 9
2	+ ·00009 7	− ·00960 3	−·00039 1	+ ·00082 7
1	+ ·00671 7	− ·35943	−·01857 9	− ·08330
0	+ ·46038	− ·82002	−·65366	+1·19699
−1	+ ·59651	− ·02285 7	+·46862	+ ·05037 5
−2	+ ·02033 9	− ·00017 9	−·00525 9	+ ·00064 8
−3	+ ·00017 6	− ·00000 1	−·00003 9	+ ·00000 7
−4	+ ·00000 2		−·00000 1	
Sum	+1·08422	−1·21224	−·20932	+1·16557

106. *Characteristic* e'^2k^2.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2m+2g$.		$2i-2m-2g$.	
4			+·00004 18	
3	−·00000 03		+·00224 16	
2	+·00000 75		+·07521 17	+·00000 14
1	+·00043 03	−·00000 08	−·53998 50	−·00079 35
0	+·09250 43	+·00101 26	+·07918 70	−·00969 97
−1	−·68816 31	−·09327 73	−·00013 90	+·00002 11
−2	+·03528 74	−·00795 57	+·00005 74	+·00000 01
−3	+·00101 97	−·00005 78	+·00000 12	
−4	+·00001 95	−·00000 05		
Sum	−·55889 47	−·10027 95	−·38338 33	−·01047 06

i.	$2i+2m-2g$.		$2i-2m+2g$.	
4	+·00000 20		−·00000 02	
3	+·00006 94		+·00001 86	
2	+·00203 74	+·00000 01	+·00116 32	
1	+·12148 35	+·00010 79	+·04889 60	−·00000 04
0	−·12571 86	−·00395 81	−·11524 84	+·00001 30
−1	+·21424 14	−·00030 39	+·12137 41	−·00037 31
−2	+·00727 26	−·00000 43	−·00002 34	−·00042 17
−3	+·00014 40		+·00002 67	−·00000 17
−4	+·00000 18		+·00000 11	
Sum	+·21953 35	−·00415 83	+·05620 77	−·00078 39

Characteristic e'^2k^2.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2m$.		$2i-2m$.	
4	−·00000 08		−·00000 89	
3	−·00002 15		−·00047 64	
2	−·00156 93	−·00000 01	−·01535 04	
1	−·11505 11	−·00000 61	+·44404 18	+·00000 55
0	−·17343 72	−·00016 14	−·13677 02	+·01266 87
−1	+·52334 25	+·11337 32	−·10334 50	+·00245 57
−2	+·02473 90	+·00102 26	−·00042 17	+·00002 21
−3	+·00058 62	+·00000 90	+·00000 55	+·00000 02
−4	+·00001 06	+·00000 01	+·00000 01	
Sum	+·25859 84	+·11423 73	+·18767 48	+·01515 22

i.	$2i+2g$.		$2i-2g$.	
4			−·00001 75	
3	−·00000 51		−·00108 34	
2	−·00024 95		−·04714 61	+·00000 19
1	−·00940 95	+·00000 12	+·29280 13	+·00080 10
0	−·03505 90	−·00082 36	−·00795 42	−·02647 03
−1	+·32065 65	+·03486 41	−·05402 71	−·00010 85
−2	−·02496 30	+·00468 96	−·00194 98	+·00000 02
−3	−·00056 64	+·00002 20	−·00004 05	
−4	−·00000 91	+·00000 02	−·00000 07	
Sum	+·25039 49	+·03875 35	+·18058 20	−·02577 57

Characteristic e'^2k^2.

Values of $\mathfrak{K}+\mathfrak{L}$.

$i.$	$2i.$	
4	+·00000 33	
3	+·00013 97	
2	+·00404 21	
1	−·22402 63	+·00011 94
0	+·33808 64	−·00148 01
−1	−·32859 84	−·05109 26
−2	−·01208 10	−·00033 08
−3	−·00024 56	−·00000 27
−4	−·00000 33	
Sum	−·22268 31	−·05278 68

Values of

$i.$	$(\eta^2k^2)_i$.	$(\eta'^2k'^2)_i$.	$(\eta^2k'^2)_i$.	$(\eta'^2k^2)_i$.
3		+·00009 7	+·00000 1	
2		+·00617 8	−·00012 8	+·00000 6
1	+·00000 7	+·17825 2	−·07136	+·00033 3
0	+·00167 8	+·02792 3	−·08335 9	+·00475 7
−1	−·38676 9	−·00001 8	+·02048 7	+·13082
−2	+·01312 8	+·00000 1	+·00026 7	·00000 0
−3	+·00008 1		+·00000 3	+·00000 2
Sum	−·37187 5	+·21243 3	−·13409	+·13592

Characteristic $e'^2\text{k}^2$.

Values of

i.	$(\eta^2kk')_i$.	$(\eta'^2kk')_i$.	$(\eta\eta'k^2)_i$.	$(\eta\eta'k'^2)_i$.	$(\eta\eta'kk')_i$.
3		−·00002 0		−·00004 6	+·00000 68
2	−·00001 9	−·00142 2	−·00000 1	−·00348 4	+·00048 87
1	−·00001 1	−·07253 2	+·00001 5	−·19279	+·02689 8
0	+·27381	−·41568	+·00101 0	−·01919 2	+·08383 2
−1	+·59306 6	−·05074 8	+·30225	−·00469 2	−·26628 1
−2	+·00284 8	−·00001 6	−·00945 8	−·00006 7	−·00121 51
−3	+·00002 4	+·00000 1	−·00004 3	−·00000 1	−·00000 96
Sum	+·86972	−·54042	+·29377	−·22027	−·15628 0

107. *Characteristic* k.^{4}

Values of $\mathfrak{K}$. For arguments $2i \pm 4g$, $2i$, $\mathfrak{L}=0$.

i.	$2i+4g$.	$2i-4g$.	$2i$.
4		+·00000 44	
3		+·00026 86	+ ·00002 20
2	+·00000 07	+·01037 01	+ ·00183 54
1	+·00005 65	−·12623 98	+ ·12460 70
0	+·00288 65	+·02178 99	−3·96849 90
−1	−·09817 71	+·00042 43	+ ·09150 01
−2	+·01326 43	+·00000 58	+ ·00077 12
−3	+·00008 29		+ ·00000 57
−4	+·00000 24		
Sum	−·08188 38	−·09337 67	−3·74975 76

Characteristic k⁴.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2g$.		$2i-2g$.	
4			− ·00000 03	
3	+ ·00000 03		− ·00003 75	
2	+ ·00008 09		+ ·00059 77	−·00000 33
1	+ ·01375 25	+·00000 02	− ·01261 91	−·00193 31
0	+1·96250 00	+·00025 03	+1·95644 07	+·03951 52
−1	+ ·01515 37	−·00427 50	− ·01125 33	+·00008 92
−2	+ ·00211 13	−·00001 36	− ·00040 44	+·00000 04
−3	+ ·00007 78	−·00000 01	− ·00000 64	
−4	+ ·00000 15			
Sum	+1·99367 80	−·00403 82	+1·93271 74	+·03766 84

Values of

i.	$(k^4)_i$.	$(k'^4)_i$.	$(k^3k')_i$.	$(kk'^3)_i$.	$(k^2k'^2)_i$.
3		+·00001 3		−·00000 6	+·00000 05
2		+·00072 4		−·00053 8	+·00002 68
1		−·05111 5	−·00003 6	−·03284	−·00012 0
0	+·00001 1	+·00171 4	−·00096 1	+·98818 1	−·99206 1
−1	+·00029 6	+·00001 5	+·02794	−·00121 8	+·06300 2
−2	+·00412 0		+·00260 9	−·00001 5	+·00004 69
−3	+·00004 2		+·00000 9		+·00000 02
Sum	+·00446 9	−·04864 9	+·02956	+·95356	−·92910 5

108. *Characteristic* $e^3\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+3c$.		$2i-3c$.	
9			−·00018 71	
7	+·00000 19		−·01000 07	−·00000 34
5	−·00002 09		−·09655 71	+·00000 30
3	−·00939 13	−·00000 27	+·04866 76	+·00008 93
1	−·63721 77	−·00036 52	+·18169 35	−·00037 34
−1	+1·05819 96	+·00042 40	−·05478 44	+·00181 04
−3	+·05883 14	+·00032 86	−·00065 86	−·00181 00
−5	−·02176 80	+·00035 12	+·00000 19	−·00003 19
−7	−·00114 07	+·00001 62		−·00000 04
−9	−·00001 96	−·00001 67		
Sum	+·44747 47	+·00073 54	+·06817 51	−·00031 64

$2i$.	$2i+c$.		$2i-c$.	
9	+·00000 04		−·00002 05	
7	−·00005 87		−·00220 26	−·00000 03
5	−·00326 16	−·00000 17	−·12122 64	−·00004 70
3	−·48587 63	−·00023 37	−·64460 95	−·00001 98
1	−·71097 95	+·00184 02	+·18169 80	−·00102 75
−1	+·19813 19	−·00148 48	−·15240 58	−·00236 47
−3	−·13854 40	−·00165 65	−·04646 56	+·00442 62
−5	−·01271 97	+·00110 69	−·00069 53	−·00109 85
−7	−·00020 42	−·00022 96	−·00000 45	−·00001 93
−9	−·00000 24	−·00000 40	−·00000 03	−·00000 03
Sum	−1·15851 41	−·00066 32	−·78593 25	−·00015 12

Characteristic $e^3\alpha$.

Values of

$2i.$	$(\epsilon^3\alpha)_i.$	$(\epsilon'^3\alpha)_i.$	$(\epsilon^2\epsilon'\alpha)_i.$	$(\epsilon\epsilon'^2\alpha)_i.$
9		−·00000 5		−·00000 1
7		−·00039 8	−·00000 2	−·00006 6
5	−·00000 3	−·00925 5	−·00023 9	−·00436 5
3	−·00027 4	−·01426	−·01591 3	−·05569 0
1	−·01911 6	+·01030 2	−·05442 6	−·10225
−1	+·07964 1	−·00130 2	+·18382	−·02624 3
−3	+·03460	−·00005 9	−·01911 2	−·00089 3
−5	−·00230 3		−·00023 8	−·00004 1
−7	−·00002 7		−·00001 0	
−9	−·00000 1			
Sum	+·09252	−·01498	+·09388	−·18955

109. *Characteristic* $e^2e'\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i.$	$2i+2c+m.$		$2i-2c-m.$		$2i+2c-m.$	
9			−·00009			
7	+ ·00003		−·00668		+ ·00004	
5	+ ·00262		−·29203	−·00027	+ ·00036	
3	+ ·16029		−·27590	+·00137	− ·03993	−·00004
1	+6·03740	−·00001	−·11737	−·00649	− ·77599	−·00262
−1	− ·42403	+·00261	+·74524	+·00422	−1·83937	+·00191
−3	− ·12488	−·04085	+·01684	+·00599	+ ·15188	+·00185
−5	− ·04515	+·00835	+·00025	+·00009	+ ·03582	−·00200
−7	− ·00078	−·00159			+ ·00071	+·00028
−9	+ ·00001	−·00003			+ ·00002	
Sum	+5·60551	−·03152	+·07026	+·00491	−2·46646	−·00062

Characteristic $e^2e'\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i-2c+m$.		$2i+m$.		$2i-m$.	
9	+·00012		+·00002		+·00002	
7	+·00688		+·00115		−·00023	
5	+·26155	+·00001	+·06615		−·03842	−·00002
3	+·39707	+·00047	+2·47134	+·00001	−1·40678	−·00179
1	−·62492	−·00995	+·89452	+·00425	−·69566	+·00653
−1	−·10050	+·01256	−·27015	−·03575	+·30308	−·01102
−3	−·00121	−·01973	−·20414	+·04986	+·32001	−·00456
−5	+·00012	−·00042	−·00349	−·01142	+·00706	+·00259
−7	+·00001	−·00001	−·00002	−·00021	+·00010	+·00004
−9						
Sum	−·06088	−·01707	+2·95538	+·00674	−1·51082	−·00823

Values of

$2i$.	$(\epsilon^2\eta\alpha)_i$.	$(\epsilon'^2\eta'\alpha)_i$.	$(\epsilon^2\eta'\alpha)_i$.	$(\epsilon'^2\eta\alpha)_i$.	$(\epsilon\epsilon'\eta\alpha)_i$.	$(\epsilon\epsilon'\eta'\alpha)_i$.
7		−·0002		+·0003		
5		−·0177		+·0129	+·0023	−·0013
3	+·0051	−·2686	−·0011	−·0879	+·1281	−·0746
1	+·2742	−·1039	−·0435	+2·1242	+1·4461	−·0768
−1	−·0294	+·0341	−1·0316	−·0010	−·1597	−3·8796
−3	+·4801	+·0005	+·4427	−·0009	−·0009	+·0135
−5	−·0003		+·0013		−·0007	+·0002
−7	−·0002		−·0001			
Sum	+·7295	−·3558	−·6323	+2·0476	+1·4152	−4·0186

110. *Characteristic* $ee'^2\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

2*i*.	2*i*+c+2m.		2*i*−c−2m.		2*i*+c−2m.	
9			+·00001 1			
7	−·00004 4		+·00026 6		+·00017 0	
5	−·00287 6		−·04865 0	−·00005 2	+·00424 1	−·00000 2
3	−·13497 9	+·00000 7	−·76647 7	−·00666 6	−·02201 2	−·00024 5
1	−2·50990 8	−·00292 0	−·62264 6	+·03178 7	+·26600 8	−·01207 2
−1	−·80061 9	+·01047 1	−·45296 4	+·01308 5	−·47218 9	+·01051 8
−3	−·15688 6	+·34620 1	−·01619 4	−·00036 4	−·06595 8	+·00408 9
−5	−·00452 7	−·05404 8	−·00031 5	−·00003 7	−·00350 9	+·00233 8
−7	+·00011 7	−·00079 1	−·00000 2	−·00000 1	−·00007 7	+·00000 5
−9		−·00001 0				
Sum	−3·60972 2	+·29891 0	−1·90698 1	+·03775 2	−·29332 6	+·00463 1

2*i*.	2*i*−c+2m.		2*i*+c.		2*i*−c.	
9					+·00004 6	
7	−·00061 0		+·00023 9		+·00360 1	
5	−·02829 2	−·00000 1	+·01488 5	+·00000 1	+·17806 3	−·00000 3
3	−·35775 3	−·00013 1	+·64255 4	+·00010 7	+2·91345 5	+·00266 3
1	−·52080 9	+·00478 9	+2·14351 8	−·01498 3	+·68113 3	+·01973 1
−1	+·06614 8	+·09080 3	+·73440 6	+·03484 2	+·40067 2	+·02618 8
−3	+·00826 5	−·13086 1	+·61312 7	−·11872 9	+·06910 6	+·04654 8
−5	+·00092 1	−·00267 4	+·02246 9	+·01657 8	+·00143 3	+·00087 0
−7	+·00003 6	−·00003 7	+·00041 5	+·00023 1	+·00003 4	+·00001 2
−9		−·00000 1	+·00000 3	+·00000 3		
Sum	−·83209 4	−·03811 3	+4·17161 6	−·08195 0	+4·24754 3	+·09600 9

Characteristic $ee'^2\alpha$.

Values of

$2i$.	$(\epsilon\eta^2\alpha)_i$.	$(\epsilon'\eta'^2\alpha)_i$.	$(\epsilon\eta'^2\alpha)_i$.	$(\epsilon'\eta^2\alpha)_i$.	$(\epsilon\eta\eta'\alpha)_i$.	$(\epsilon'\eta\eta'\alpha)_i$.
5		−·0023		−·0011	+·0004	+·0071
3	−·0050	−·1649	+·0001	−·0297	+·0211	+·2706
1	−·1865	−·1208	+·0051	−·1912	+·1708	−·0926
−1	−·2071	−·0297	−·0445	+·1006	+·4363	+·0461
−3	+·4312	−·0004	−·0004	−·0126	−·0316	+·0067
−5	−·0061		+·0001	−·0001	+·0023	+·0001
Sum	+·0265	−·3181	−·0396	−·1341	+·5993	+·2380

III. *Characteristic* $e'^3\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+3$m.		$2i-3$m.	
9			+·00001 4	
7	+·00000 9		+·00037 1	
5	+·00043 4		+·00820 4	−·00000 4
3	+·00474 3	+·00006 4	−·19909 9	−·00062 0
1	−·29874 9	−·00700 1	−·04918 3	−·03895 6
−1	−·00992 9	+·05054 9	−·09107 7	+·00285 0
−3	−·03587 3	−·60765 5	+·00026 9	−·00063 4
−5	+·00204 6	−·00855 2	+·00004 9	−·00000 8
−7	+·00006 6	−·00010 0	+·00000 1	
−9	+·00000 2	−·00000 1		
Sum	−·33725 1	−·57269 6	−·33045 1	−·03737 2

Characteristic $e'^3\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+\mathrm{m}$.		$2i-\mathrm{m}$.	
9			+ ·00000 9	
7	−·00015 9		+ ·00040 8	
5	−·00858 7		+ ·02460 2	+·00000 1
3	−·26123 8	−·00002 3	+ ·83332 0	+·00004 5
1	−·01739 6	−·00566 6	+ ·25130 2	−·00662 8
−1	+·08868 9	−·03773 4	− ·00639 3	+·01028 3
−3	+·14305 5	+·31427 9	− ·04100 1	−·00738 4
−5	+·00288 4	+·00347 8	− ·00104 5	−·00021 0
−7	+·00004 4	+·00004 0	− ·00001 8	−·00000 4
−9	+·00000 1	+·00000 1		
Sum	−·05270 7	+·27437 5	+1·06118 4	−·00389 7

Values of

$2i$.	$(\eta^3\alpha)_i$.	$(\eta'^3\alpha)_i$.	$(\eta^2\eta'\alpha)_i$.	$(\eta\eta'^2\alpha)_i$.
5	+·00001	+·00036	−·00028	+·00072
3	−·00012	+·00972	−·01429	+·03354
1	−·0723	+·3042	−·3287	−·2531
−1	−·7294	+·0898	+·7869	+·9756
−3	−·1709	−·00002	+·08589	−·00436
−5	−·00046		+·00028	−·00004
Sum	−·9732	+·4041	+·5298	+·7524

112. *Characteristic* ek²α.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+c+2g$.		$2i-c-2g$.	
9			−·00002 63	
7	− ·00000 06		−·00232 93	
5	+ ·00000 13		−·12694 76	−·00003 62
3	− ·00042 98	+·00000 01	+·18390 29	−·00016 14
1	− ·05518 61	+·00002 60	+·23086 82	+·00590 31
−1	+1·98177 62	−·00800 76	−·54487 76	−·00573 30
−3	+ ·21544 85	+·01729 16	−·00453 24	−·00009 31
−5	− ·03239 84	−·00025 38	+·00001 95	−·00000 10
−7	− ·00064 34	−·00017 96	+·00000 06	
−9	− ·00000 68	−·00000 19		
Sum	+2·10856 09	+·00887 48	−·26392 20	−·00012 16

$2i$.	$2i+c-2g$.		$2i-c+2g$.	
9	−·00000 09			
7	−·00013 36		− ·00000 25	
5	−·01084 86	−·00000 15	− ·00030 68	
3	−·51891 28	+·00014 81	− ·01792 58	−·00000 56
1	−·14053 51	−·00594 66	+1·70964 27	+·00054 14
−1	+·54234 01	+·01310 49	− ·18777 01	+·02116 52
−3	−·14735 54	+·01124 11	− ·11929 11	−·02572 49
−5	−·00168 14	+·00004 72	− ·00276 70	+·00047 81
−7	−·00000 68	+·00000 02	− ·00003 48	−·00000 26
−9			− ·00000 01	−·00000 01
Sum	−·27713 45	+·01859 34	+1·38154 45	−·00354 85

Characteristic ek²α.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+c$.		$2i-c$.	
9			+ ·00000 34	
7	+ ·00001 07		+ ·00023 93	+·00000 03
5	+ ·00126 86	−·00000 10	+ ·02399 07	+·00002 16
3	+ ·11958 89	−·00009 19	+1·50798 22	−·00409 44
1	+6·75069 59	+·01948 61	−1·78455 48	−·02157 33
−1	−1·93792 53	−·04226 85	+1·07322 87	−·02465 41
−3	+ ·19026 33	−·02071 99	− ·02962 91	+·00703 99
−5	− ·00889 43	+·00496 60	− ·00036 56	+·00013 23
−7	− ·00012 71	+·00004 56	+ ·00000 03	+·00000 15
−9	− ·00000 10	+·00000 04		
Sum	+5·11487 97	−·03858 32	+ ·79089 51	−·04312 62

Values of

$2i$.	$(\epsilon k^2 \alpha)_i$.	$(\epsilon' k'^2 \alpha)_i$.	$(\epsilon k'^2 \alpha)_i$.	$(\epsilon' k^2 \alpha)_i$.
7		−·00012 7	−·00000 5	
5		−·01181 1	−·00052 0	+·00000 4
3	+·00002 3	−·08718 3	−·04041 2	+·00064 2
1	+·00187 9	−·03664 1	−·20220	+·11156 3
−1	+·15508 4	−·04612 8	+·17870 1	+·10583
−3	+·15427 2	−·00017 7	−·01258 5	−·04372 5
−5	−·00719 0		−·00006 3	−·00007 6
−7	−·00005 1		−·00000 1	−·00000 1
Sum	+·30401 7	−·18206 7	−·07709	+·17424

Characteristic $ek^2\alpha$.

Values of

$2i$.	$(\epsilon kk'\alpha)_i$.	$(\epsilon' kk'\alpha)_i$.
7	+ ·00000 1	+ ·00001 5
5	+ ·00006 3	+ ·00166 1
3	+ ·00669 3	+ ·16219 6
1	+ ·56588 5	+1·44113
−1	−2·22530	− ·03941 1
−3	− ·10800 4	− ·00373 9
−5	− ·00080 9	− ·00001 5
−7	− ·00000 5	
Sum	−1·76148	+1·56184

113. *Characteristic* $e'k^2\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+m+2g$.		$2i-m-2g$.	
9			− ·00000 71	
7	+ ·00000 18		− ·00049 35	
5	+ ·00017 33		− ·03515 46	−·00000 07
3	+ ·01544 48	+·00000 02	− ·07602 70	−·00097 56
1	+1·06559 78	+·00041 62	− ·28776 46	−·00955 78
−1	− ·60593 45	−·04399 71	+3·05205 66	−·01139 56
−3	+ ·00478 07	−·11860 46	+ ·05553 97	+·00002 02
−5	− ·01154 32	−·00649 08	+ ·00070 44	+·00000 06
−7	− ·00014 63	−·00006 41	+ ·00000 78	
−9	− ·00000 21	−·00000 05		
Sum	+ ·46837 23	−·16874 07	+2·70886 17	−·02190 89

Characteristic $e'k^2\alpha$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$2i$.	$2i+m-2g$.		$2i-m+2g$.	
9	+·00000 20			
7	+·00021 14		+ ·00000 18	
5	+·01621 97	−·00000 34	+ ·00007 23	
3	+·49466 56	+·00069 13	+ ·00293 93	+·00000 05
1	−·97149 67	−·00578 50	+ ·07585 28	−·00011 26
−1	−·02020 43	−·04771 53	−2·92221 25	−·01749 37
−3	+·00345 46	−·00060 99	+ ·18175 84	+·02886 88
−5	+·00035 14	−·00000 62	+ ·00704 50	+·00308 27
−7	+·00000 91		+ ·00009 40	+·00001 74
−9	+·00000 02		+ ·00000 06	+·00000 01
Sum	−·47678 70	−·05342 85	−2·65444 83	+·01436 32

$2i$.	$2i+m$.		$2i-m$.	
9	− ·00000 04		− ·00000 10	
7	− ·00001 65		+ ·00003 17	
5	− ·00150 99		+ ·00417 52	+·00000 02
3	− ·09289 49	+·00002 48	+ ·33260 11	+·00008 40
1	−1·26453 15	−·00687 78	+1·00154 29	+·01019 64
−1	+ ·43988 60	+·19119 70	− ·44129 44	+·02820 24
−3	− ·08255 25	+·11614 20	+ ·19227 95	−·03913 82
−5	− ·00097 39	+·00120 59	+ ·00317 00	−·00025 30
−7	+ ·00000 62	+·00001 11	+ ·00003 92	−·00000 19
−9	− ·00000 02		+ ·00000 04	
Sum	−1·00258 76	+·30170 30	+1·09254 46	−·00091 01

Characteristic $e'k^2\alpha$.

Values of

$2i$.	$(\eta k^2\alpha)_i$.	$(\eta' k'^2\alpha)_i$.	$(\eta k'^2\alpha)_i$.	$(\eta' k^2\alpha)_i$.
7		−·00002 9	+·00001 1	
5		−·00335 8	+·00142 1	+·00000 1
3	+·00004 4	−·34996	+·17656	+·00008 6
1	+·00358 8	−·68238	−·02358	+·00346 5
−1	+·10476	+·50351 5	−·01345 1	−·48005
−3	+·74417	+·00284 3	+·00013 7	−·20583
−5	−·00328 7	+·00001 9	+·00000 9	+·00157 1
−7	−·00001 1			+·00000 5
Sum	+·84926	−·52935	+·14111	−·68075

$2i$.	$(\eta kk'\alpha)_i$.	$(\eta' kk'\alpha)_i$.
7	− ·00000 2	+ ·00000 2
5	− ·00026 7	+ ·00022 0
3	− ·03252 0	+ ·02120 2
1	−4·31577	+ ·71841
−1	−1·32620	+12·28216
−3	− ·00148 2	+ ·03487 2
−5	− ·00001 7	+ ·00016 4
−7		+ ·00000 1
Sum	−5·67626	+13·05703

114. *Characteristic* $e^2\alpha^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	$2i+2c$.		$2i-2c$.		$2i$.	
4			+·00028		+ ·00007	
3	+·00018		+·01627	−·00004	+ ·00281	
2	+·00754		+·29412	−·00033	+ ·12784	−·00035
1	+·26891	−·00061	−·12528	−·00745	+1·02879	+·00473
0	−·37623	+·03189	−·06085	+·00066	− ·30835	−·00958
−1	−·14252	−·01204	+·02427	+·00084	+ ·21573	−·01470
−2	+·06237	−·03041	+·00086	−·00632	+ ·01220	+·02769
−3	+·00174	+·00495	+·00003	−·00008	+ ·00031	−·00245
−4	+·00004	−·00024			+ ·00001	−·00003
Sum	−·17797	−·00646	+·14970	−·01272	+1·07941	+·00531

Values of

i.	$(\epsilon^2\alpha^2)_i$.	$(\epsilon'^2\alpha^2)_i$.	$(\epsilon\epsilon'\alpha^2)_i$.
3		+·00069	+·00008
2	+·00018	+·0330	+·00442
1	+·00829	+·035	+·08709
0	−·02731	−·0025	−·0795
−1	−·095	+·00077	+·0172
−2	−·0249	−·00020	+·00291
−3	+·00054		−·00008
Sum	−·138	+·067	+·0321

115. *Characteristic* ee'/a^2.

Values of $\mathfrak{K}+\mathfrak{L}$.

i.	2*i*+c+m.		2*i*−c−m.	
4			+·00021	
3	− ·00009		+·00753	−·00001
2	− ·02560		+·26215	−·00263
1	−2·75577	−·00278	+·20795	+·00899
0	+1·34099	−·00605	+·41844	+·00925
−1	+ ·08386	+·01829	−·34740	+·01662
−2	+ ·04510	+·24946	−·00299	+·00849
−3	+ ·00099	−·02085	−·00002	+·00007
−4	+ ·00003	−·00024		
Sum	−1·31049	+·23783	+·54587	+·04078

i.	2*i*+c−m.		2*i*−c+m.	
4	+·00003		− ·00006	
3	+·00125		− ·00753	
2	+·03440	−·00003	− ·72035	−·00053
1	+·23283	−·00644	−1·13600	+·00531
0	+·61656	+·02227	+ ·22661	+·02298
−1	−·40173	+·00163	+ ·03323	+·05310
−2	−·10185	−·05084	+ ·00479	−·07805
−3	−·00100	+·00181	+ ·00021	−·00102
−4	−·00001	+·00002	+ ·00001	−·00001
Sum	+·38048	−·03158	−1·59909	+·00178

Characteristic $ee'\alpha^2$.

Values of

$i.$	$(\epsilon\eta\alpha^2)_i.$	$(\epsilon'\eta'\alpha^2)_i.$	$(\epsilon\eta'\alpha^2)_i.$	$(\epsilon'\eta\alpha^2)_i.$
3	−·00001	+·00026	+·00003	−·00037
2	−·00131	+·00751	+·00106	−·03513
1	−·12949	+·1213	+·01164	−·1270
0	+·1225	+·2751	+·2975	−·5059
−1	−·1899	−·01338	−·2103	+·01300
−2	+·0662	+·00047	−·01635	−·00417
−3	−·00121		+·00010	−·00002
Sum	−·1332	+·3913	+·0837	−·6596

116. *Characteristic* $k^2\alpha^2$.

Values of $\mathfrak{K}+\mathfrak{L}$.

$i.$	$2i+2g.$		$2i-2g.$		$2i.$	
3	+·00001		+·00037		+·00004	
2	+·00047		+·15316	+·00009	−·00055	+·00002
1	+·01697	+·00027	−·22860	+·00117	−1·50841	−·00180
0	−·46493	−·03704	−·07083	−·10214	+·74163	+·10091
−1	−·23351	−·03121	+·08806	−·02647	−·10082	+·13475
−2	+·04289	−·03335	+·00279	−·00018	+·00400	+·02164
−3	+·00017	−·00009	+·00005		+·00012	+·00014
Sum	−·63793	−·10142	−·05500	−·12753	−·86399	+·25566

Values of

$i.$	$(k^2\alpha^2)_i.$	$(k'^2\alpha^2)_i.$	$(kk'\alpha^2)_i.$
3		+·00015	−·00001
2		+·02250	−·00110
1	+·00008	+·632	−·17168
0	−·03354	−·0508	+·2109
−1	−·623	+·00535	+·2299
−2	−·0300	+·00009	+·00274
−3	−·00002		+·00001
Sum	−·686	+·609	+·2708

117. *Characteristic* ke^3.

Values of $K+L$. For arguments $2i+\mathrm{g}\pm 2\mathrm{c}$, $L=0$.

i.	$2i+\mathrm{g}+3\mathrm{c}$.	$2i+\mathrm{g}-3\mathrm{c}$.
5		+·00000 56
4	+·00000 03	+·00022 74
3	+·00001 44	+·00533 39
2	+·00074 35	+·00402 52
1	+·03004 59	−·01541 20
0	+·71696 43	−·07503 40
−1	−·10081 77	−·02229 53
−2	+·00022 10	−·00089 00
−3	−·00021 24	−·00002 15
−4	−·00015 28	−·00000 03
−5	−·00000 65	
Sum	+·64680 00	−·10406 10

i.	$2i+\mathrm{g}+\mathrm{c}$.		$2i+\mathrm{g}-\mathrm{c}$.	
5	+·00000 01		+·00000 16	
4	+·00000 83		+·00008 52	
3	+·00043 36		+·00340 21	
2	+·01736 50	−·00000 07	+·07930 55	−·00000 12
1	+·40822 14	−·00007 16	+·00652 77	−·00011 17
0	−·01854 41	−·00635 53	+·06525 69	+·00001 00
−1	+·00110 98	+·00022 09	−·02001 07	−·00001 61
−2	−·00204 37	−·00003 89	−·01211 33	−·00000 02
−3	−·00229 06	−·00000 04	−·00050 44	
−4	−·00009 76		−·00001 26	
−5	−·00000 25		−·00000 02	
Sum	+·40415 97	−·00624 60	+·12193 78	−·00011 92

Characteristic ke^3.

Values of

i.	$(k\epsilon^3)_i$.	$(k\epsilon'^3)_i$.	$(k\epsilon^2\epsilon')_i$.	$(k\epsilon\epsilon'^2)_i$.
4		+·00000 70	+·00000 01	+·00000 14
3	+·00000 01	+·00038 82	+·00000 68	+·00009 92
2	+·00001 12	+·00180 89	+·00048 15	+·00531 28
1	+·00079 37	+·01349 47	+·02534 49	+·00201 07
0	+·04139 88	−·02230 57	−·00708 02	−·05535 33
−1	−·02439 31	−·00142 20	−·00106 86	−·00749 18
−2	+·00007 65	−·00002 50	−·00094 57	−·00083 02
−3	−·00012 54	−·00000 03	−·00016 94	−·00001 49
−4	−·00001 23		−·00000 30	−·00000 02
Sum	+·01774 95	−·00805 42	+·01656 64	−·05626 63

118. *Characteristic* ke^2e'.

Values of $K+L$.

i.	$2i+g+2c+m$.		$2i+g-2c-m$.	
5			+·00000 66	
4	−·00000 04		+·00030 69	
3	−·00002 30		+·01032 92	+·00000 05
2	−·00107 37	+·00000 01	+·16457 95	+·00004 19
1	−·03569 68	+·00001 10	−·01782 22	+·00016 40
0	−·54560 36	+·00091 97	+·00562 27	−·00079 83
−1	−·05701 99	−·00043 15	+·04641 16	−·00003 02
−2	+·00702 83	−·00001 58	+·00202 05	−·00000 03
−3	−·00663 19	−·00000 13	+·00005 09	
−4	−·00036 53		+·00000 09	
−5	−·00001 01			
Sum	−·63939 64	+·00048 22	+·21150 66	−·00062 24

Characteristic ke²e′.

Values of $K+L$.

i.	$2i+g+2c-m$.		$2i+g-2c+m$.	
5			−·00000 08	
4	+·00000 26		−·00004 42	
3	+·00012 68		−·00149 82	+·00000 05
2	+·00539 69	+·00000 01	−·02391 83	+·00004 19
1	+·14580 39	+·00001 10	−·00160 56	+·00016 40
0	+·65773 97	+·00091 97	+·00971 18	−·00079 83
−1	−·00337 70	−·00043 15	−·08389 85	−·00003 02
−2	+·00043 36	−·00001 58	−·00626 45	−·00000 03
−3	+·00177 32	−·00000 13	−·00019 66	
−4	+·00007 97		−·00000 40	
−5	+·00000 20			
Sum	+·80798 14	+·00048 22	−·10771 89	−·00062 24

i.	$2i+g+m$.		$2i+g-m$.	
5	−·00000 02		+ ·00000 12	
4	−·00000 91		+ ·00005 78	
3	−·00043 65		+ ·00260 01	
2	−·01468 09	+·00000 47	+ ·08054 40	+·00000 29
1	−·23463 84	+·00040 37	+ ·96859 73	+·00025 48
0	−·02250 64	+·00011 90	− ·01639 85	−·00012 32
−1	+·01512 91	−·00065 50	− ·00262 58	+·00018 93
−2	−·04967 40	−·00001 82	+ ·01797 53	−·00001 06
−3	−·00305 52	−·00000 02	+ ·00079 78	−·00000 01
−4	−·00009 00		+ ·00002 02	
−5	−·00000 17		+ ·00000 01	
Sum	−·30996 33	−·00014 60	+1·05156 95	+·00031 31

Characteristic $\mathrm{k}e^2e'$.

Values of

i.	$(k\epsilon^2\eta)_i$.	$(k\epsilon^2\eta')_i$.	$(k\epsilon^2\eta')_i$.	$(k\epsilon'^2\eta)_i$.
4		+·00000 69		−·00000 09
3	−·00000 03	+·00047 35	+·00000 16	−·00006 38
2	−·00002 10	+·02356 09	+·00010 98	−·00297 65
1	−·00134 79	+·03846 7	+·00579 68	+·01507 7
0	−·05587 76	+·04524 8	+·07543 81	−·04015 9
−1	−·10870 51	+·00541 35	−·02061 5	−·01098 67
−2	−·00801 92	+·00008 30	−·00037 8	−·00027 16
−3	−·00111 32	+·00000 11	+·00025 36	−·00000 42
−4	−·00001 83		+·00000 37	−·00000 01
Sum	−·17510 26	+·11325 4	+·06061 1	−·03938 6

i.	$(k\epsilon\epsilon'\eta)_i$.	$(k\epsilon\epsilon'\eta')_i$.
4	−·00000 01	+·00000 07
3	−·00000 89	+·00005 55
2	−·00058 91	+·00344 09
1	−·02666 26	+·12360 04
0	−·12700 5	+·08781 9
−1	−·02647 80	+·00687 6
−2	−·00730 21	+·00231 70
−3	−·00014 20	+·00003 48
−4	−·00000 20	+·00000 04
Sum	−·18819 0	+·22414 5

119. *Characteristic* kee$'^2$.

Values of $K+L$.

i.	$2i+g+c+2m$.		$2i+g-c-2m$.	
5			+·00000 21	
4	+·00000 02		+·00012 17	
3	+·00000 72		+·00499 31	+·00000 02
2	+·00020 64	−·00000 01	+·12823 94	+·00002 61
1	+·00011 43	+·00000 86	+·78746 00	+·00234 71
0	−·17808 97	+·00295 39	−·17852 33	−·00867 68
−1	+·71766 93	−·00484 03	+·01447 01	−·00000 22
−2	−·05048 72	−·00012 45	−·00026 52	+·00000 02
−3	−·00595 51	−·00000 10	−·00001 76	
−4	−·00020 16		−·00000 04	
−5	−·00000 48			
Sum	+·48325 90	−·00200 34	+·75647 99	−·00630 54

i.	$2i+g+c-2m$.		$2i+g-c+2m$.	
5	+·00000 02			
4	+·00000 94		+·00000 01	
3	+·00046 12		−·00006 97	
2	+·01699 66	+·00000 10	−·00721 63	+·00000 16
1	+·33938 80	+·00009 10	−·35159 54	+·00061 65
0	+·30913 00	+·00473 20	+·31108 88	−·01333 33
−1	−·33037 02	−·00144 85	−·12455 46	−·00037 07
−2	+·00733 70	+·00000 07	−·01967 10	−·00000 40
−3	+·00010 31	+·00000 01	−·00075 20	
−4	−·00000 05		−·00001 81	
−5	−·00000 01		−·00000 01	
Sum	+·34305 47	+·00337 63	−·19278 83	−·01308 99

Characteristic kee'^2.

Values of $K+L$.

$i.$	$2i+g+c.$		$2i+g-c.$	
5			$-\cdot00000\ 05$	
4	$-\cdot00000\ 29$		$-\cdot00003\ 45$	
3	$-\cdot00014\ 54$		$-\cdot00140\ 70$	$-\cdot00000\ 08$
2	$-\cdot00554\ 54$	$+\cdot00000\ 44$	$-\cdot03566\ 80$	$-\cdot00007\ 79$
1	$-\cdot11838\ 75$	$+\cdot00042\ 44$	$-\cdot20612\ 34$	$-\cdot00712\ 07$
0	$-\cdot05326\ 65$	$+\cdot03416\ 32$	$-\cdot03830\ 00$	$-\cdot00675\ 32$
-1	$-\cdot13926\ 03$	$-\cdot00502\ 70$	$+\cdot10602\ 30$	$-\cdot00151\ 32$
-2	$+\cdot03413\ 52$	$+\cdot00010\ 36$	$+\cdot01041\ 88$	$-\cdot00001\ 72$
-3	$+\cdot00264\ 53$	$+\cdot00000\ 13$	$+\cdot00033\ 81$	$-\cdot00000\ 01$
-4	$+\cdot00007\ 99$		$+\cdot00000\ 72$	
-5	$+\cdot00000\ 16$			
Sum	$-\cdot27974\ 60$	$+\cdot02966\ 99$	$-\cdot16474\ 63$	$-\cdot01548\ 31$

Values of

$i.$	$(k\epsilon\eta^2)_i.$	$(k\epsilon'\eta'^2)_i.$	$(k\epsilon\eta'^2)_i.$	$(k\epsilon'\eta^2)_i.$
4		$+\cdot00000\ 21$	$+\cdot00000\ 01$	
3	$+\cdot00000\ 01$	$+\cdot00015\ 61$	$+\cdot00000\ 75$	$-\cdot00000\ 24$
2	$+\cdot00000\ 52$	$+\cdot00971\ 71$	$+\cdot00049\ 78$	$-\cdot00052\ 32$
1	$-\cdot00002\ 93$	$+\cdot35173\ 91$	$+\cdot02306\ 46$	$-\cdot09968\ 00$
0	$-\cdot04371\ 56$	$+\cdot15803\ 06$	$+\cdot11337\ 20$	$-\cdot25777\ 09$
-1	$-\cdot66115\ 99$	$+\cdot00480\ 33$	$+\cdot28041\ 72$	$-\cdot05958\ 97$
-2	$-\cdot03574\ 62$	$-\cdot00001\ 15$	$+\cdot00383\ 22$	$-\cdot00152\ 11$
-3	$-\cdot00052\ 45$	$-\cdot00000\ 05$	$+\cdot00001\ 26$	$-\cdot00002\ 38$
-4	$-\cdot00000\ 69$			$-\cdot00000\ 03$
Sum	$-\cdot74117\ 71$	$+\cdot52443\ 63$	$+\cdot42120\ 40$	$-\cdot41911\ 14$

Characteristic kee$'^2$.

Values of

i.	$(k\epsilon\eta\eta')_i$.	$(k\epsilon'\eta\eta')_i$.
4		$-$·00000 06
3	$-$·00000 23	$-$·00004 14
2	$-$·00015 34	$-$·00245 63
1	$-$·00732 98	$-$·07397 30
0	$-$·00507 19	$+$·03863 62
-1	$+$·12496 93	$+$·03793 55
-2	$+$·01613 76	$+$·00073 98
-3	$+$·00021 08	$+$·00001 02
-4	$+$·00000 26	$+$·00000 01
Sum	$+$·12876 29	$+$·00085 05

120. *Characteristic* ke$'^3$.

Values of $K+L$.

i.	$2i+g+3m$.		$2i+g-3m$.	
5			$+$·00000 03	
4			$+$·00001 51	
3	$-$·00000 03		$+$·00070 49	
2	$+$·00000 03		$+$·02390 12	$+$·00000 36
1	$-$·00016 08	$+$·00001 88	$+$·39028 72	$+$·00034 06
0	$-$·01185 45	$+$·02137 21	$-$·02187 35	$+$·02201 69
-1	$+$·39014 38	$-$·00489 89	$-$·00064 00	$+$·00003 96
-2	$-$·02391 05	$-$·00002 03	$-$·00019 99	$+$·00000 02
-3	$-$·00120 16	$-$·00000 01	$-$·00000 23	
-4	$-$·00003 10			
-5	$-$·00000 05			
Sum	$+$·35298 49	$+$·01647 16	$+$·39219 30	$+$·02240 09

Characteristic ke$'^3$.

Values of $K+L$.

i.	$2i+g+m$.		$2i+g-m$.	
5				
4	+·00000 05		−·00000 64	
3	+·00002 55		−·00029 19	
2	+·00062 54	+·00000 05	−·00972 22	−·00000 04
1	+·00128 41	+·00008 89	−·15175 42	−·00000 49
0	−·00357 95	+·01132 25	+·00011 85	+·01084 69
−1	−·14427 67	−·00318 17	+·00047 58	+·00029 32
−2	+·01680 23	−·00002 14	−·00094 04	+·00000 45
−3	+·00071 34	−·00000 01	−·00006 78	+·00000 01
−4	+·00001 71		−·00000 20	
−5	+·00000 03			
Sum	−·12838 76	+·00820 87	−·16219 06	+·01113 94

Values of

i.	$(k\eta^3)_i$.	$(k\eta'^3)_i$.	$(k\eta^2\eta')$.	$(k\eta\eta'^2)_i$.
4		+·00000 02		−·00000 01
3		+·00001 59	+·00000 05	−·00000 63
2		+·00110 53	+·00002 50	−·00041 80
1	−·00001 49	+·05659 12	+·00025 06	−·01944 00
0	+·00482 66	−·00172 9	+·06378 1	−·06330 2
−1	−·53057 50	−·00381 2	+·30385 23	+·00018 0
−2	−·00513 58	−·00002 82	+·00300 92	−·00012 08
−3	−·00006 46	−·00000 01	+·00003 56	−·00000 29
−4	−·00000 08		+·00000 04	
Sum	−·53096 45	+·05214 3	+·37095 5	−·08311 0

121. *Characteristic* k^3e.

Values of $K+L$. For arguments $2i+3g\pm c$, $L=0$.

i.	$2i+3g+c$.	$2i+3g-c$.
5		
4		
3	+·00000 04	− ·00000 43
2	+·00001 74	− ·00041 18
1	+·00087 34	− ·03237 90
0	+·02556 39	−1·65684 02
−1	−·27846 00	− ·11611 13
−2	−·00185 73	+ ·00073 33
−3	−·00022 02	− ·00001 73
−4	−·00001 68	− ·00000 10
−5	−·00000 06	
Sum	−·25409 98	−1·80503 16

i.	$2i+g+c$.		$2i+g-c$.	
5				
4	−·00000 02		− ·00000 65	
3	−·00002 16	−·00000 01	− ·00036 08	+·00000 16
2	−·00100 69	−·00000 57	− ·01471 73	+·00015 30
1	−·02309 88	−·00055 81	− ·24570 64	+·01382 37
0	+·86505 05	−·04955 90	+2·50685 95	−·00124 00
−1	−·09633 83	+·00172 26	+ ·10206 41	+·00199 63
−2	−·00055 86	−·00030 34	+ ·00127 02	+·00002 21
−3	−·00016 69	−·00000 33	+ ·00000 58	+·00000 02
−4	−·00000 52			
−5	+·00000 01			
Sum	+·74385 41	−·04870 70	+2·34940 86	+·01475 69

Characteristic k^3e.

Values of

i.	$(k^3\epsilon)_i$.	$(k^3\epsilon')_i$.	$(k^2k'\epsilon)_i$.	$(k^2k'\epsilon')_i$
4				− ·00000 01
3		−·00000 01	−·00000 03	− ·00001 11
2	+·00000 03	−·00001 35	−·00002 72	− ·00106 94
1	+·00002 27	−·00233 85	−·00128 45	− ·08994 42
0	+·00140 98	−·46095 52	+·23505 59	−2·12659 97
−1	−·06570 41	+·10654 96	+·07945 50	+ ·02789 93
−2	+·00250 93	+·00095 26	+·00007 65	+ ·00009 36
−3	−·00012 15	+·00000 06	−·00001 17	+ ·00000 02
−4	−·00000 15		−·00000 02	
Sum	−·06188 50	−·35580 45	+·31326 35	−2·18963 14

122. *Characteristic* k^3e'.

Values of $K+L$.

i.	$2i+3g+m$.		$2i+3g-m$.	
4				
3			+·00000 09	
2	−·00000 04		+·00005 63	
1	+·00020 12	+·00000 01	+·00212 35	+·00000 01
0	+·02002 74	+·00001 65	+·02223 28	+·00001 65
−1	−·10032 17	−·00068 38	+·02116 77	−·00068 38
−2	+·00835 45	−·00001 56	−·00348 15	−·00001 56
−3	−·00022 44	−·00000 01	+·00007 67	−·00000 01
−4	−·00000 92		+·00000 22	
Sum	−·07197 26	−·00068 29	+·04217 86	−·00068 29

Characteristic k^3e'.

Values of $K+L$.

i.	$2i+\mathrm{g}+\mathrm{m}$.		$2i+\mathrm{g}-\mathrm{m}$.	
4	+·00000 04		−·00000 19	
3	+·00001 83		−·00010 25	
2	+·00090 10	−·00000 08	−·00457 80	+·00000 40
1	+·02712 05	−·00007 43	−·10787 11	+·00030 28
0	+·04871 79	−·00030 12	+·04461 77	+·00031 18
−1	−·01092 69	+·00136 02	+·00213 57	−·00077 72
−2	−·00329 91	+·00001 45	+·00108 00	−·00000 49
−3	−·00011 59	+·00000 02	+·00002 95	
−4	−·00000 26		+·00000 08	
Sum	+·06241 36	+·00099 86	−·06468 98	−·00016 35

Values of

i.	$(k^3\eta)_i$.	$(k^3\eta')_i$.	$(k^2k'\eta)_i$.	$(k^2k'\eta')_i$.
4				
3			+·00000 04	−·00000 22
2		+·00000 11	+·00003 83	−·00020 05
1	+·00000 74	+·00008 48	+·00343 98	−·01413 28
0	+·00180 29	+·00264 45	+·26780 3	−·26499 7
−1	−·16631 44	+·10195 2	+·01283 48	+·01063 3
−2	−·00845 96	+·00444 82	−·00045 07	+·00015 08
−3	−·00005 95	+·00002 14	−·00000 54	+·00000 14
−4	−·00000 05	+·00000 01	−·00000 01	
Sum	−·17302 37	+·10915 2	+·28366 0	−·26854 7

123. *Characteristic* $ke^2\alpha$.

Values of $K+L$.

$2i$.	$2i+g+2e$.		$2i+g-2e$.	
9			−·00000 95	
7	+·00000 10		−·00075 30	+·00000 14
5	−·00005 78	+·00000 03	−·03408 84	+·00010 98
3	−·00924 98	+·00003 19	−·14174 34	−·00052 08
1	−·50561 36	+·00244 75	+·23592 66	+·00565 45
−1	+·33265 15	−·00237 97	+·08570 37	−·00163 11
−3	−·11496 98	+·00092 44	+·02136 26	−·00008 48
−5	+·00857 86	−·00003 05	+·00044 80	−·00000 10
−7	+·00130 57	−·00000 38	+·00000 48	
−9	+·00003 01			
Sum	−·28732 41	+·00099 01	+·16685 14	+·00352 80

$2i$.	$2i+g$.	
9	−·00000 02	
7	−·00005 82	+·00000 01
5	−·00553 48	+·00001 32
3	−·26471 64	+·00101 68
1	−·30003 95	−·00456 62
−1	−·26242 77	+·00689 56
−3	+·05329 61	−·00021 48
−5	+·01064 96	−·00003 40
−7	+·00023 81	−·00000 05
−9	+·00000 29	
Sum	−·76859 01	+·00311 02

Characteristic ke²α.

Values of

$2i$.	$(k\epsilon^2 a)_i$.	$(k^2\epsilon'^2 a)_i$.	$(k\epsilon\epsilon' a)_i$.
9		−·00000 02	
7		−·00002 31	−·00000 10
5	−·00000 10	−·00241 69	−·00016 04
3	−·00025 49	−·05586 70	−·01707 71
1	−·03006 16	−·20534 29	−·09528 85
−1	+·08743 57	+·02665 51	+·21904 30
−3	+·10025 96	+·00141 20	+·02244 01
−5	+·00501 11	+·00001 32	+·00077 23
−7	+·00010 56	+·00000 01	+·00000 74
−9	+·00000 10		+·00000 01
Sum	+·16249 55	−·23556 97	+·12973 59

124. *Characteristic* kee′α.

Values of $K+L$.

$2i$.	$2i+g+c+m$.		$2i+g-c-m$.	
9			+·00000 4	
7	+ ·00003 0		−·00006 1	
5	+ ·00213 9		−·01062 8	+·00007 6
3	+ ·11381 3	−·00000 3	−·40554 7	+·00611 0
1	+3·68451 4	+·00250 2	+·03619 5	−·04405 2
−1	− ·01515 8	−·02278 5	+·00627 8	−·00129 8
−3	− ·16887 4	−·00097 0	−·12646 4	+·00011 1
−5	+ ·02886 5	−·00030 8	−·00363 7	+·00000 2
−7	+ ·00074 8	−·00000 3	−·00006 7	
−9	+ ·00001 0			
Sum	+3·64608 7	−·02156 7	−·50392 7	−·03905 1

Characteristic keε′α.

Values of $K+L$.

$2i$.	$2i+g+c-m$.		$2i+g-c+m$.	
9			+ ·00000 7	
7	+·00001 9		+ ·00044 1	
5	+·00038 6	+·00000 2	+ ·02328 2	−·00000 2
3	−·01159 3	+·00028 5	+ ·74244 1	+·00043 3
1	−·29154 5	+·01677 6	+ ·00857 2	−·02096 1
−1	+·01202 9	−·01348 2	− ·06503 7	−·01387 3
−3	+·03803 6	+·00033 8	+ ·07677 2	−·00103 3
−5	−·02610 6	+·00002 9	+ ·00203 5	−·00001 2
−7	−·00076 4		+ ·00001 8	
−9	−·00001 4			
Sum	−·27955 2	+·00394 8	+ ·78853 1	−·03544 8

Values of

$2i$.	$(k\epsilon\eta\alpha)_i$.	$(k\epsilon'\eta'\alpha)_i$.	$(k\epsilon\eta'\alpha)_i$.	$(k\epsilon'\eta\alpha)_i$.
7				+·00000 9
5	+·00004 4	−·00048 5	+·00000 8	+·00098 2
3	+·00452 5	−·05375 8	−·00047 6	+·08783 6
1	+·39617 9	+·03317	−·03291 9	−·4212
−1	−·08295	−·1332	+·1682	+·23485
−3	+·28615 6	−·01555 7	−·12480	+·01082 7
−5	+·00498 4	−·00015 7	−·00374 8	+·00009 6
−7	+·00003 9	−·00000 2	−·00003 6	
Sum	+·60898	−·1700	+·0062	−·0866

125. *Characteristic* $\mathrm{k}e'^2\alpha$.

Values of $K+L$.

$2i$.	$2i+g+2m$.		$2i+g-2m$.		$2i+g$.	
9			+·00000 3		+·00000 1	
7	−·00001 6		+·00007 3		+·00009 3	
5	−·00094 7		+·00215 8	+·00001 2	+·00520 9	−·00000 1
3	−·03544 6	+·00004 7	+·02615 5	+·00103 4	+·18452 3	+·00004 4
1	−·28833 7	+·01564 3	+·20918 6	+·06701 3	−·10637 3	+·02244 8
−1	+·14056 4	+·06545 3	−·22082 6	−·00171 1	−·09188 9	+·00184 8
−3	+·11342 9	−·00751 3	+·06191 8	+·00002 3	−·16626 7	+·00094 4
−5	+·00284 1	−·00005 6	+·00239 3	−·00000 1	−·00906 2	+·00001 1
−7	−·00000 3		+·00004 8		−·00019 9	
−9					−·00000 4	
Sum	−·06791 5	+·07357 4	+·08110 8	+·06637 0	−·18396 8	+·02529 4

Values of

$2i$.	$(k\eta^2\alpha)_i$.	$(k\eta'^2\alpha)_i$.	$(k\eta\eta'\alpha)_i$.
7		+·00000 1	+·00000 2
5	−·00002 6	+·00006 5	+·00014 9
3	−·00215 4	+·00201 2	+·01187 9
1	−·07106 2	+·11048 3	−·02612 1
−1	−·18423 6	+·18843 7	+·07789 9
−3	+·05465 0	+·02052 8	−·06604 4
−5	+·00026 7	+·00017 3	−·00069 8
−7		+·00000 2	−·00000 6
Sum	−·20256 1	+·32170 1	−·00294 0

126. *Characteristic* $k^3\alpha$.

Values of $K+L$.

$2i$.	$2i+3g$.		$2i+g$.	
9			+ ·00000 01	
7	−·00000 01		+ ·00000 88	
5	+·00000 65		+ ·00077 21	−·00000 07
3	+·00040 11	+·00000 05	+ ·05639 87	−·00008 47
1	+·01996 76	+·00005 98	+2·48539 91	−·01078 67
−1	+·74206 40	+·00810 84	+1·66872 82	−·01733 51
−3	−·18534 79	−·00058 52	+ ·12295 52	−·00038 94
−5	+·00258 76	−·00003 09	+ ·00276 64	−·00000 23
−7	+·00010 36	−·00000 02	+ ·00004 10	
−9	+·00000 16		+ ·00000 07	
Sum	+·57978 40	+·00755 24	+4·33707 03	−·02859 89

Values of

$2i$.	$(k^3\alpha)_i$.	$(k^2k'\alpha)_i$.
7		+ ·00000 02
5	+·00000 01	+ ·00002 57
3	+·00001 17	+ ·00424 96
1	+·00132 75	+ ·77503 01
−1	+·19243 08	−1·41930 03
−3	+·16488 20	+ ·04209 68
−5	+·00249 91	+ ·00022 04
−7	+·00001 21	+ ·00000 14
Sum	+·36116 33	− ·59767 61

127. *Characteristic* $ke\alpha^2$.

Values of $K+L$.

i.	$2i+g+c$.		$2i+g-c$.	
4	+·00000 1		+·00002 5	
3	+·00013 8		+·00105 4	+·00001 5
2	+·00567 2	+·00007 6	+·03302 6	+·00152 3
1	+·14568 3	+·00810 0	+·27137 4	−·03529 1
0	−·03785 2	+·00671 5	−·04866 7	−·02091 6
−1	+·20850 9	−·02037 7	−·07390 1	−·00240 4
−2	−·03552 0	+·00131 5	−·00479 0	−·00026 5
−3	−·00131 5	−·00004 8	−·00014 0	−·00000 3
−4	−·00002 9		−·00000 4	
Sum	+·28528 7	−·00421 9	+·17797 7	−·05734 1

Values of

i.	$(ke\alpha^2)_i$.	$(ke'\alpha^2)_i$.
3	+·00000 2	+·00003 2
2	+·00016 0	+·00238 4
1	+·00954 3	+·08233 9
0	−·00952 1	+·05848 6
−1	−·16281 9	−·02730 1
−2	−·01633 8	−·00036 7
−3	−·00011 7	−·00000 4
Sum	−·17909 0	+·11556 9

128. *Characteristic* $ke'\alpha^2$.

Values of $K+L$.

i.	$2i+g+m$.		$2i+g-m$.	
4	−·00000 1		+·00001 3	
3	−·00008 4		+·00053 9	+·00000 4
2	−·01155 0	+·00005 4	+·01512 1	+·00048 8
1	−·84138 2	+·01252 8	+·04220 3	+·05264 0
0	−·00439 4	+·03115 9	−·00709 1	+·01538 3
−1	+·02933 5	+·05394 8	−·02860 8	−·00535 1
−2	+·00117 6	−·00267 6	+·03479 3	+·00000 1
−3	−·00031 2	−·00002 0	+·00058 0	
−4	−·00001 2		+·00000 5	
Sum	−·82722 4	+·09499 3	+·05755 5	+·06316 5

Values of

i.	$(k\eta\alpha^2)_i$.	$(k\eta'\alpha^2)_i$.
3	−·00000 2	+·00001 2
2	−·00049 6	+·00066 0
1	−·09352 2	+·01198 7
0	+·12870	−·06371
−1	−·17592 4	+·18589
−2	−·00054 0	+·00476 3
−3	−·00001 6	+·00002 8
Sum	−·14180	+·13963

Haverford College, Pa., U.S.A.:
1900 *May* 14.

(To be continued.)

Theory of the Motion of the Moon; containing a New Calculation of the Expressions for the Coordinates of the Moon in terms of the Time. By ERNEST W. BROWN, M.A., Sc.D., F.R.S.

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PART IV. CHAPTERS VII.–IX.

THE previous parts of this memoir have been published in the *Memoirs of the Royal Astronomical Society* under the same title in 1897, 1899, 1900. The solution of the problem undertaken—the motion of the Moon as disturbed by the Sun supposed to move in a fixed elliptic orbit—is completed in the present part.

It was stated in the introduction to Part I. that the main object in view was a new and accurate calculation of every coefficient in longitude, latitude, and parallax which is as great as one-hundredth of a second of arc, the result not to be in error by more than this amount. So far as I am able to see this plan has been carried out. A careful examination of the magnitude of the coefficients which would arise with characteristics higher than those calculated here, and a comparison with the results of DELAUNAY and HANSEN, seem to show that no characteristics which would give coefficients so great as 0″·01 have been omitted. There are possibly four or five terms whose characteristics are of the sixth order, which approach 0″·01 quite closely. These omissions are, however, quite unimportant from a practical point of view. In a comparison between theory and observation a few such terms produce nothing sensible in the differences; it is only when the number of them is great that any effect is shown.

The coefficients in longitude and latitude in every characteristic calculated have actually been found to 0″·001, and in parallax to 0″·0001. In the former case a large number of coefficients between 0″·01 and 0″·001 are included; and, in fact, it is not difficult to see that there are comparatively few coefficients lying between 0″·002 and 0″·01 which are not included in the tables at the end of Chapter IX. Thus the theory, for purposes of comparison with observation, is considerably more accurate than was contemplated in the original plan. Moreover, a similar remark applies to the coefficients in parallax lying between 0″·0002 and 0″·001; thus the parallax of

the Moon can certainly be found theoretically from the new table within $0''\cdot01$, so far as the solar perturbations are concerned.

To complete the whole problem of the lunar motion, inequalities arising from other sources have still to be considered. These consist of the very minute terms arising from the parts of the solar disturbing function which are noted in Chap. I. § 4, the terms arising from the figure of the Earth, and perhaps from that of the Moon, the indirect planetary inequalities and the direct planetary inequalities. The last is the only set which presents serious difficulties at the present time, and an investigation of them has already been started. Hill's work * on the inequalities produced by the figure of the Earth probably needs but little supplementing, while a new method † for investigating indirect planetary inequalities should render the task of calculating their coefficients comparatively easy.

An important question which cannot be left aside is that of the accuracy of the computations by which the results have been obtained. It appears unlikely that the problem will be again completely solved in the near future, and some assurance is needed that the new coefficients, especially where they differ from those of Hansen or Delaunay, are the correct ones. Fortunately the three theories are so entirely independent in their methods that agreement between them all amounts to practical certainty. The differences between Hansen and Delaunay have given rise to much discussion in the past. In general my theory confirms the results of Delaunay in these cases; the coefficients in which all three theories differ are those which are difficult to determine owing to the presence of very small divisors. The older methods, which approximate along powers of m, are theoretically less likely to be correct than a method which approximates along powers of the other parameters where the convergence is quite rapid. This, however, is a rather different question than that of the actual accuracy of the numerical work. For the latter very numerous tests have been used, covering almost every detail, as well as large masses of calculations. These tests are discussed below in Chap. VII. Sect. (iv), as well as in other papers to which reference will there be found.

To return to the special part of the work now published. For the terms of the fifth order in u the homogeneous equations were used for the first time. In spite of the fact that this was a change of method involving much extra work the computations were thus kept within reasonable bounds; the expansion of $\kappa u/r^3$ would have nearly doubled the actual work done. The non-homogeneous equation was still used for z, as most of the multiplications had been obtained in calculating the fourth order terms. But for the sixth order in both u, z the non-homogeneous equations were used, and the work was much less than had been expected.

The final steps consisted in the transformation to polar coordinates, the change of

* *Washington Astronomical Papers*, vol. iii. pt. 2, 1891.

† E. W. Brown, *Trans. Amer. Math. Soc.*, vol. vi. (1905).

the arbitrary constants to the DELAUNAY system, and the insertion of their numerical values to reduce the results to seconds of arc. For the first of these the formulæ were found to permit of such arrangements that much previous work could be usefully utilised. In the final reduction to numbers, values for the constants were used which were neither those of HANSEN nor those of DELAUNAY. A selection was made from modern determinations which I believe will be found to be very close to the more accurate values to be found when a thorough comparison of the completed theory with the observations has been undertaken.

Owing mainly to the complicated character of the work which is embodied in the results below I have been obliged to do much more actual calculation myself than heretofore. All computations which could with advantage be turned over to a computer have again been done by Mr. IRA I. STERNER, A.M.* His speed and accuracy have been fully maintained, and have contributed in no small degree to an earlier conclusion of the work than I had hoped. He has in all spent some three thousand hours on these calculations, extended over seven and a half years; my own share I estimate at five or six thousand hours since the work was begun on a complete plan in 1895.

The following is the table of contents for the whole memoir :—

Chapter I.—General Development of the Theory.

Section (i). An investigation of the disturbing function used, with the necessary corrections.

Section (ii). The two forms of the equations of motion.

Section (iii). Development of the disturbing function according to powers of $1/a'$, z, e'.

Section (iv). The form of the solution. The general system of notation adopted to represent the coefficients, arguments, &c.

Section (v). Method of solution. Preparation of the equations of motion.

Section (vi). Exact definitions of the arbitrary constants used in the theory.

Section (vii). Methods used for the solution of the equations of condition satisfied by the coefficients. The long and short period terms which give rise to small divisors. Manner of obtaining the higher parts of the motions of the perigee and node.

Section (viii). Details concerning the numerical calculations and the methods used to verify them.

Section (ix). Transformation to polar coordinates.

Chapter II.—Terms of zero order. Numerical results.

Chapter III.—Numerical results for terms of the first order.

Chapter IV.—Numerical results for terms of the second order.

Chapter V.—Terms of the third order.

* The expense has been met by grants from the Government Grant Committee of the Royal Society.

Errata will be found at the ends of Chapters V., IX.

CHAPTER VII.

Terms of the Fifth Order.

Section (i). *Preparation of the Equations for u, s.*

129. *The Homogeneous Equations for u, s.* As stated above, it was found necessary, in order to keep the calculations within reasonable limits, to change the method from the non-homogeneous to the homogeneous form of the equations, as far as u_5, s_5 were concerned.

Equations (6), (7) of Chap. I. are those to be used. They may be written—

$$
\begin{aligned}
0 = \Phi &\equiv \mathrm{f} + \tfrac{9}{4}\mathrm{m}^2\mathrm{L}' + \delta\mathrm{f} \\
&\equiv D^2(us+z^2) - Du\,.\,Ds - (Dz)^2 - 2\mathrm{m}(uDs - sDu) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2z^2 + 3\omega_2 + 4\omega_3 - D^{-1}(D'\omega_2 + D'\omega_3) \qquad (1)\\
0 = \Psi &\equiv \mathrm{f}' + \tfrac{3}{2}\mathrm{m}^2D^{-1}\Lambda' + D^{-1}\delta(D\mathrm{f}') \\
&\equiv uDs - sDu - 2\mathrm{m}us + D^{-1}\left\{\tfrac{3}{2}\mathrm{m}^2(u^2-s^2) - s\frac{\partial\omega_2}{\partial s} - s\frac{\partial\omega_3}{\partial s} + u\frac{\partial\omega_2}{\partial u} + u\frac{\partial\omega_3}{\partial u}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)
\end{aligned}
$$

where the constants of integration are omitted, since they contain only characteristics of even order, and q takes only the values 2, 3, since characteristics containing α^2 are neglected. The new symbols must be defined; in all cases, unless stated otherwise, suffixes represent the orders of the characteristics present in the functions to which they are attached.

The majority of the indices of ζ contain c, g. Now, to the order considered here,

$$\mathrm{c} = \mathrm{c}_0 + \mathrm{c}_2 + \mathrm{c}_4, \quad \mathrm{g} = \mathrm{g}_0 + \mathrm{g}_2 + \mathrm{g}_4,$$

and therefore the operations D, D^2, D^{-1} introduce parts of c, g other than c_0, g_0; these parts must be separated. Put

$$
\begin{aligned}
\mathrm{f} &\equiv D^2(us+z^2) - Du\,.\,Ds - (Dz)^2 - 2\mathrm{m}(uDs - sDu) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2z^2,\\
\mathrm{f}' &\equiv uDs - sDu - 2\mathrm{m}us + \tfrac{3}{2}\mathrm{m}^2D^{-1}(u^2-s^2),\\
\mathrm{L}' &\equiv \frac{4}{9\mathrm{m}^2}\{3\omega_2 + 4\omega_3 - D^{-1}(D'\omega_2 + D'\omega_3)\},\\
\Lambda' &\equiv \frac{2}{3\mathrm{m}^2}\left\{u\frac{\partial\omega_2}{\partial u} + u\frac{\partial\omega_3}{\partial u} - s\frac{\partial\omega_2}{\partial s} - s\frac{\partial\omega_3}{\partial s}\right\},
\end{aligned}
$$

where c_0, g_0 are substituted for c, g in the coefficients when the operations D, D^2, D^{-1} are performed. Hence

$\delta\mathrm{f}$, $\delta(D\mathrm{f}')$ are respectively the parts due to $\mathrm{c}-\mathrm{c}_0$, $\mathrm{g}-\mathrm{g}_0$ in

$$D^2(us+z^2) - Du\,.\,Ds - (Dz)^2 - 2\mathrm{m}(uDs - sDu) - D^{-1}(D'\omega_2 + D'\omega_3),$$
$$D(uDs - sDu - 2\mathrm{m}us).$$

Denote the unknown coefficients of $\zeta^{\pm(2i+\tau)}$ in $u_5\zeta^{-1}/\mathrm{a}$ by λ_i, λ'_{-i}. The equations $\Phi=0$, $\Psi=0$ are linear with respect to all these coefficients, since terms of order higher

than 5 are neglected and those of lower order have been found. It is necessary to put the equations into forms convenient for calculation; the method is implicitly contained in § 33. It will be seen from that section that if we equate to zero the coefficients of $\zeta^{2i+\tau}$ in

$$\frac{1}{D^2-1-2m+\frac{1}{2}m^2}[\Phi+(1+2m)\Psi]=0,\quad \frac{1}{D}\left[\Psi+\frac{2(1+m)}{D^2-1-2m+\frac{1}{2}m^2}\{\Phi+(1+2m)\Psi\}\right]=0, \quad \dots \quad (3)$$

the terms of principal importance, involving the unknowns in the left-hand members, are $a(\lambda_i+\lambda'_{-i})$, $-a(\lambda_i-\lambda'_{-i})$, respectively. This fact serves as a guide for the arrangement of the equations which now follows.

130. Let

$$\vartheta u \equiv D(u\zeta^{-1})=\zeta^{-1}(Du-u),\ \vartheta s \equiv D(s\zeta)=\zeta(Ds+s),$$
$$F = D^2-1-2m+\tfrac{1}{2}m^2,$$

so that ϑu_0, ϑs_0 are divisible by m^2 and

$$F^{-1}\zeta^{2i+\tau}=\frac{\zeta^{2i+\tau}}{(2i+\tau)^2-1-2m+\frac{1}{2}m^2},$$

then it will be found that

$$\Phi+(1+2m)\Psi \equiv F(us+z^2)-\vartheta u\,.\,\vartheta s-(Dz)^2+(1+2m-\tfrac{7}{2}m^2)z^2+\delta f$$
$$+\tfrac{9}{4}m^2(u^2+s^2+L')+\tfrac{3}{2}m^2(1+2m)D^{-1}\left\{u^2-s^2+\Lambda'+\frac{2}{3m^2}\delta(Df')\right\} \quad \dots \quad \dots \quad (4)$$

$$\Psi \equiv u\vartheta s-s\vartheta u-2(1+m)us+\frac{3m^2}{2}D^{-1}\left\{u^2-s^2+\Lambda'+\frac{2}{3m^2}\delta(Df')\right\} \quad \dots \quad \dots \quad (5)$$

It is to be noticed that all terms except those in ω_3 are homogeneous products of the second order with respect to u, s, and their derivatives.

There are terms whose characteristics are of orders 0, 1, 2, 3, 4, 5 in u, s, z. Let

$$u_a,\ s_a,\ z_a\ (a=5,\ 4,\ 3) \text{ and } u_b,\ s_b,\ z_b\ (b=5-a)$$

distinguish the orders, it being noted that $z_0=0$ and therefore that z_5 is not present. Then

$$u\vartheta s-s\vartheta u=\sum_{a=3}^{a=5}\left[2(u_a\vartheta s_b-s_a\vartheta u_b)-D(u_a s_b-s_a u_b)\right] \qquad (b=5-a),$$

and we easily obtain from equation (5),

$$D^{-1}\Psi=-\sum_{a=3}^{a=5}(u_a s_b-u_b s_a)+D^{-1}\left[2\sum_{a=3}^{a=5}(u_a\vartheta s_b-s_a\vartheta u_b)-2(1+m)us+\tfrac{3}{2}m^2D^{-1}\left\{u^2-s^2+\Lambda'+\frac{2}{3m^2}\delta(Df')\right\}\right. \quad (6)$$

Equations (3) are immediately derivable from (4), (6).

131. *Forms for Computation.*—In the following formulæ the bar over any expression, as usual, means that $1/\zeta$ has been put for ζ. The sign Σ, denoting summation for values of a, is omitted for the sake of brevity when no misunderstanding of the meaning can arise.

Put

$$G = u_a u_b + \overline{u_a u_b} + \tfrac{1}{2}L' \qquad \ldots \qquad (7)$$

$$H = u_a u_b - \overline{u_a u_b} + \tfrac{1}{2}\Lambda' + \frac{1}{3\mathrm{m}^2}\delta(D\mathfrak{f}') \ldots \qquad (8)$$

$$A = -\vartheta u_a \,.\, \vartheta s_b - \overline{\vartheta u_a \,.\, \vartheta s_b} - 2Dz_a \,.\, Dz_b + 2z_a z_b + (2\mathrm{m} - \tfrac{7}{2}\mathrm{m}^2)2z_a z_b + \delta\mathfrak{f} + \tfrac{9}{2}\mathrm{m}^2 G + 3\mathrm{m}^2(1+2\mathrm{m})D^{-1}H \quad (9)$$

$$B = u_a \vartheta s_b + \overline{u_a \vartheta s_b} + 2(1+\mathrm{m})\tfrac{1}{2}F^{-1}A + 2z_a z_b + \mathrm{m} \,.\, 2z_a z_b + 3\mathrm{m}^2(1+2\mathrm{m})D^{-1}H \div 2(1+2\mathrm{m}) \qquad \ldots \quad (10)$$

Then the values of λ_i, λ'_{-i} are obtained by equating to zero the coefficients of $\zeta^{2i+\tau}$ in the equations

$$u_a s_b - D^{-1}B + z_a z_b + \tfrac{1}{2}F^{-1}A = 0 \quad \ldots \quad (11)$$

$$\overline{u_a s_b} + D^{-1}B + z_a z_b + \tfrac{1}{2}F^{-1}A = 0 \quad \ldots \quad (12)$$

It will then be found that the principal coefficients of λ_i, λ'_{-i} derived from the left-hand member of (11) are respectively a, 0, and from (12) 0, a, while all the other unknowns have coefficients small compared with a.

132. *Method for Approximation.*—Denote the difference between the ith and $(i-1)$th approximation to any function Q by $Q^{(i)}$. The first approximations to G, H, A, B are obtained by neglecting all the unknowns in these four functions. For G, H have m^2 as a factor, while u_5, ϑu_5 occur elsewhere in A, B only when multiplied by ϑs_0, which also has m^2 as a factor. In (11), (12) all the unknowns except those we are considering are multiplied by terms which also contain the same factor. Hence $G^{(1)}$, $H^{(1)}$, $A^{(1)}$, $B^{(1)}$, $\lambda_i^{(1)}$, $\lambda'_{-i}{}^{(1)}$ are obtained from the known terms in the equations of § 131.

The second approximation is obtained by substituting $u_5^{(1)}$ for u_5 in the various formulæ. Then $u_5^{(2)}$ is determined from

$$G^{(2)} = u_5^{(1)}u_0 + \overline{u_5^{(1)}u_0}, \quad H^{(2)} = u_5^{(1)}u_0 - \overline{u_5^{(1)}u_0},$$

$$A^{(2)} = -\vartheta u_5^{(1)} \,.\, \vartheta u_0 - \overline{\vartheta u_5^{(1)} \,.\, \vartheta u_0} + \tfrac{9}{2}\mathrm{m}^2 G^{(2)} + 3\mathrm{m}^2(1+2\mathrm{m})D^{-1}H^{(2)},$$

$$B^{(2)} = u_5^{(1)}\vartheta s_0 + \overline{u_5^{(1)}\vartheta s_0} + 2(1+\mathrm{m})\tfrac{1}{2}F^{-1}A^{(2)} + 3\mathrm{m}^2(1+2\mathrm{m})D^{-1}H^{(2)} \div 2(1+2\mathrm{m}),$$

$$u_5^{(2)} \,.\, \mathrm{a}\zeta^{-1} + u_5^{(1)}(s_0 - \mathrm{a}\zeta^{-1}) - D^{-1}B^{(2)} + \tfrac{1}{2}F^{-1}A^{(2)} = 0,$$

$$\overline{u_5^{(2)} \,.\, \mathrm{a}\zeta^{-1}} + \overline{u_5^{(1)}(s_0 - \mathrm{a}\zeta^{-1})} + D^{-1}B^{(2)} + \tfrac{1}{2}F^{-1}A^{(2)} = 0.$$

The further approximations proceed in a similar manner.

Exceptions to this method will only occur when the divisors introduced by F^{-1}, D^{-1} are such as to render it useless, or very tedious, owing to the number of approximations required. For such cases a special method is devised in § 138, below.

133. *Development of* L′, Λ′.—These functions depend solely on the terms in ω_2, ω_3, where (§ 8)

$$\omega_2 = \tfrac{3}{4}\mathrm{m}^2(u^2\mathfrak{a}_2 + s^2\bar{\mathfrak{a}}_2) + \tfrac{1}{2}\mathrm{m}^2 us\mathfrak{b}_2 - \mathrm{m}^2 z^2\mathfrak{b}_2,$$

$$a'\omega_3 = \tfrac{5}{8}\mathrm{m}^2(u^3\mathfrak{a}_3 + s^3\bar{\mathfrak{a}}_3) + \tfrac{3}{8}\mathrm{m}^2(u^2s\mathfrak{c}_3 + us^2\bar{\mathfrak{c}}_3) - \tfrac{3}{2}\mathrm{m}^2 z^2(u\mathfrak{c}_3 + s\bar{\mathfrak{c}}_3).$$

Hence

$$\begin{aligned} L' = {} & u^2\mathfrak{a}_2 + s^2\bar{\mathfrak{a}}_2 + \tfrac{2}{3}us\mathfrak{b}_2 - \tfrac{4}{3}z^2\mathfrak{b}_2 + D^{-1}\{-\tfrac{1}{3}(u^2D\mathfrak{a}_2 + s^2D\bar{\mathfrak{a}}_2) - \tfrac{2}{9}usD\mathfrak{b}_2 + \tfrac{4}{9}z^2D\mathfrak{b}_2\} \\ & + \frac{1}{a'}\{\tfrac{10}{9}(u^3\mathfrak{a}_3 + s^3\bar{\mathfrak{a}}_3) + \tfrac{2}{3}(u^2s\mathfrak{c}_3 + us^2\bar{\mathfrak{c}}_3) - \tfrac{8}{3}z^2(u\mathfrak{c}_3 + s\bar{\mathfrak{c}}_3)\} - \frac{1}{a'}D^{-1}\{\tfrac{5}{18}(u^3D\mathfrak{a}_3 + s^3D\bar{\mathfrak{a}}_3) \\ & + \tfrac{1}{6}(u^2sD\mathfrak{c}_3 + us^2D\bar{\mathfrak{c}}_3) - \tfrac{2}{3}z^2(uD\mathfrak{c}_3 + sD\bar{\mathfrak{c}}_3)\} \end{aligned}$$

$$\Lambda' = u^2\mathfrak{a}_2 - s^2\bar{\mathfrak{a}}_2 + \frac{1}{a'}\{\tfrac{5}{4}(u^3\mathfrak{a}_3 - s^3\bar{\mathfrak{a}}_3) + \tfrac{1}{4}(u^2s\mathfrak{c}_3 - s^2u\bar{\mathfrak{c}}_3) - z^2(u\mathfrak{c}_3 - s\bar{\mathfrak{c}}_3)\}.$$

The terms in these expressions are all known, for the German letters having suffix 2 contain e' as a factor, and those having the suffix 3 have α as a factor. These are expanded (§ 9) in powers of e', while $u^2=(u^2)_4+(u^2)_3+(u^2)_2+(u^2)_1+(u^2)_0$, &c. The terms whose characteristics are of order 5 are chosen out to obtain L′, Λ′.

134. *Development of* δf, $\delta(Df')$.—The calculation of these functions is troublesome and a more detailed exposition is advisable. We have (§ 129)

$$\delta\mathrm{f} = \mathrm{c}_2\frac{\partial \mathrm{f}_3}{\partial \mathrm{c}} + \mathrm{g}_2\frac{\partial \mathrm{f}_3}{\partial \mathrm{g}} + \tfrac{1}{2}\mathrm{c}_2{}^2\frac{\partial^2 \mathrm{f}_1}{\partial \mathrm{c}^2} + \mathrm{c}_4\frac{\partial \mathrm{f}_1}{\partial \mathrm{c}},$$

where the partial derivatives refer only to the *coefficients* in f, and c_0, g_0 are put for c, g. The terms containing $\mathrm{c}_2\mathrm{g}_2$, $\mathrm{g}_2{}^2$, g_4 are absent, since f_1 does not contain g. The new parts of c contained in c_4 are as yet unknown; they will be determined in Section (ii), and meanwhile it may be noted that they are only present in the coefficients of $\zeta^{2i\pm c}$.

The expression for δf arises (§ 129) from

$$\delta\mathrm{f} = \delta\{D^2(us) - \vartheta u\,.\,\vartheta s - (1+2\mathrm{m})(u\vartheta s - s\vartheta u) + \mathrm{Q}\},$$

where

$$\mathrm{Q} = D^2(z^2) - (Dz)^2 - D^{-1}(D'\omega_2 + D'\omega_3). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

To find $\partial\mathrm{f}_3/\partial\mathrm{c}$ put

$$u_a\zeta^{-1} = \Sigma\mu_q\zeta^q, \quad u_b\zeta^{-1} = \Sigma\mu_{q_1}\zeta^{q_1},$$

where

$$a = 3,\ 2\ ;\ b = 3-a\ ;\ q - q_1 = 2i+\tau = p\ ;$$

and where the summation signs refer to all the terms present. Then

$$\frac{\partial \mathrm{f}_3}{\partial \mathrm{c}} = \Sigma\zeta^p\frac{\partial}{\partial \mathrm{c}}\left[\{p^2+qq_1+(1+2\mathrm{m})(q+q_1)\}\mu_q\mu_{q_1} + \{p^2+qq_1-(1+2\mathrm{m})(q+q_1)\}\mu_{-q}\mu_{-q_1}\right] + \frac{\partial \mathrm{Q}_3}{\partial \mathrm{c}}$$

$$= \Sigma\zeta^p\left[\left(2p\frac{\partial p}{\partial \mathrm{c}} + q\frac{\partial q_1}{\partial \mathrm{c}} + q_1\frac{\partial q}{\partial \mathrm{c}}\right)(\mu_q\mu_{q_1} + \mu_{-q}\mu_{-q_1}) + (1+2\mathrm{m})\left(\frac{\partial q}{\partial \mathrm{c}} + \frac{\partial q_1}{\partial \mathrm{c}}\right)(\mu_q\mu_{q_1} - \mu_{-q}\mu_{-q_1})\right] + \frac{\partial \mathrm{Q}_3}{\partial \mathrm{c}},$$

the summation sign referring to all possible values of p, q, q_1.

Since $p = q - q_1$, the first factor may be written

$$p\frac{\partial p}{\partial \mathrm{c}} - p\frac{\partial q_1}{\partial \mathrm{c}} + q\left(\frac{\partial q}{\partial \mathrm{c}} + \frac{\partial q_1}{\partial \mathrm{c}}\right).$$

Also

$$\begin{aligned}\Sigma\zeta^p q(\mu_q\mu_{q_1} + \mu_{-q}\mu_{-q_1}) &= \Sigma(s_b\vartheta u_a + u_b\vartheta s_a),\\ \Sigma\zeta^p(\mu_q\mu_{q_1} \pm \mu_{-q}\mu_{q_1}) &= \Sigma(s_b u_a \pm u_b s_a).\end{aligned} \qquad \begin{cases} a = 3,\ 2\ ; \\ b = 3-a. \end{cases}$$

Omitting the summation sign for the two values of a, as in § 131, and putting

$$\frac{\mathrm{E}}{\mathrm{E}'} = u_a s_b \pm \overline{u_a s_b}, \qquad \frac{\mathrm{F}}{\mathrm{F}'} = s_b\vartheta u_a \pm \overline{s_b\vartheta u_a},$$

the results can be symbolically expressed in the form

$$\frac{\partial \mathrm{f}_3}{\partial \mathrm{c}} = \Sigma'\left[\left(\frac{\partial p}{\partial \mathrm{c}} - \frac{\partial q_1}{\partial \mathrm{c}}\right)D\mathrm{E} + (1+2\mathrm{m})\left(\frac{\partial q}{\partial \mathrm{c}} + \frac{\partial q_1}{\partial \mathrm{c}}\right)\mathrm{E}' + \left(\frac{\partial q}{\partial \mathrm{c}} + \frac{\partial q_1}{\partial \mathrm{c}}\right)\mathrm{F}'\right] + \frac{\partial \mathrm{Q}_3}{\partial \mathrm{c}},$$

in which the sign Σ' denotes that in the formation of E the proper coefficient arising from $\partial p/\partial\mathrm{c} - \partial q_1/\partial\mathrm{c}$ is to be attached to the corresponding term in E, and similarly for

the other two terms. The above algebraical expression appears more complicated than it is in actual calculation, for the derivatives of p, q, q_1 are integers less than 4.

By a quite similar investigation and with the same notation we obtain

$$\frac{\partial}{\partial c}(D\mathrm{f}_3') = -2\Sigma'\left[(1+\mathrm{m})\frac{\partial p}{\partial c}\mathrm{E} + \frac{\partial q_1}{\partial c}D\mathrm{E}' + \frac{\partial p}{\partial c}\mathrm{F}\right].$$

The derivatives with respect to g are obtained in replacing $\partial/\partial c$ by $\partial/\partial g$ in the above formulæ.

The same formulæ serve for $\partial \mathrm{f}_1/\partial c$, $\partial(D\mathrm{f}_1')/\partial c$, but they simplify. We have $Q_1 = 0$, $a = 1$, $b = 0$, $q = p = 2i \pm c$, $q_1 = 0$; the only terms present are those of characteristic e.

Finally, in a similar manner, we obtain

$$\frac{\partial^2 \mathrm{f}_1}{\partial c^2} = 2(u_c s_0 + s_c u_0), \quad \frac{\partial^2(D\mathrm{f}_1')}{\partial c^2} = -2(u_e s_0 - s_e u_0).$$

135. *Development of* $\partial Q_3/\partial c$, $\partial Q_3/\partial g$.—Differentiating (13) in the previous §, we obtain from the only possible combination of a, b,

$$\begin{aligned}\frac{\partial Q_3}{\partial c} &= 2\frac{\partial}{\partial c}[D^2(z_{kc}z_k) - D(z_{kc}Dz_k) + z_{kc}D^2 z_k] - \frac{\partial}{\partial c}D^{-1}(D'\omega_2 + D'\omega_3) \\ &= \Sigma'\frac{\partial p}{\partial c}[4D(z_{kc}z_k) - 2z_{kc}Dz_k + D^{-2}(D'\omega_2 + D'\omega_3)],\end{aligned}$$

which, in connection with § 133, gives the required formula.

For the derivative with respect to g of the first two terms of Q we have

$$2\frac{\partial}{\partial g}[D^2(z_1 z_2) - Dz_1 \,.\, Dz_2] = 4\Sigma'\frac{\partial p}{\partial g}D(z_1 z_2) - 2Dz_2\frac{\partial}{\partial g}Dz_1 - 2Dz_1\frac{\partial}{\partial g}Dz_2.$$

Now z only contains ζ in the combination $\zeta^{2i+\tau} - \zeta^{-2i-\tau}$. Let z' denote the value of z when this expression is replaced by $\zeta^{2i+\tau} + \zeta^{-2i-\tau}$. Then, since only the first multiple of g occurs in z_2, z_1,

$$\frac{\partial}{\partial g}(Dz_2) = z_2', \quad \frac{\partial}{\partial g}(Dz_1) = z_1', \quad z_1'Dz_2 + z_2'Dz_1 = z_1'Dz_2 - z_1 Dz_2' + D(z_1 z_2').$$

Hence

$$\frac{\partial Q_3}{\partial g} = 2\left[2\Sigma'\frac{\partial p}{\partial g}D(z_1 z_2) - z_1'Dz_2 + z_1 Dz_2' - D(z_1 z_2')\right] + \Sigma'\frac{\partial p}{\partial g}D^{-2}(D'\omega_2 + D'\omega_3).$$

Section (ii). *The New Parts of* c. *Terms with Small Divisors.*

136. *Determination of* c_4.—In those terms of the fifth order containing $\zeta^{2i\pm c}$ a new part, c_4, of c arises for determination, and one of the unknown coefficients is indeterminate; the definition of the latter (§ 25) is such that the coefficients of $\zeta^{\pm c}$ in $u_5\zeta^{-1}$ are to be equal. One of the unknowns is thus replaced by c_4, but the linear character of the equations is retained. The formulæ for finding c_4 and the coefficient of ζ^c must,

however, be quite differently formed from those for the other unknowns. The main reason for this is that in the case of $\zeta^{\pm c}$ we have

$$F^{-1}\zeta^{\pm c} = \zeta^{\pm c} \div (c_0^2 - 1 - 2m + \tfrac{1}{2}m^2),$$

since c_0 is put for c after the operation F^{-1} (§ 130) ; the divisor is therefore very small. If all powers of m had been included in F the divisor would have been zero and approximation impossible. Hence the formulæ must be so arranged that F^{-1} is not present.

Equations (10), (11), (12) of § 131 may be written

$$B = B' + (1+m)F^{-1}A, \qquad B = \tfrac{1}{2}D\Sigma(u_a s_b - s_a u_b), \qquad A = -F(us + z^2),$$

giving

$$A - \frac{1}{2+2m} FD\Sigma(u_a s_b - s_a u_b) + \frac{FB'}{1+m} = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (14)$$

$$-us - z^2 + \frac{B'}{1+m} - \frac{1}{2+2m} D\Sigma(u_a s_b - s_a u_b) = 0 \quad \dots \quad \dots \quad \dots \quad (15)$$

equations which are free from the operator F^{-1}, since A, B′ do not contain it. These two quantities contain c_4. To isolate it put (see equations (9), (10))

$$A = A' + c_4\left\{\frac{\partial f_1}{\partial c} + (1+2m)D^{-1}\frac{\partial}{\partial c}(Df_1')\right\},$$

$$B' = B'' + \tfrac{1}{2}c_4 D^{-1}\frac{\partial}{\partial c}(Df_1').$$

Substituting in (14), (15) we obtain

$$c_4\left[\frac{\partial f_1}{\partial c} + \left\{1 + 2m + \frac{F}{2+2m}\right\}D^{-1}\frac{\partial}{\partial c}(Df_1')\right] = -A' + \frac{F}{1+m}\{\tfrac{1}{2}D\Sigma(u_a s_b - s_a u_b) - B''\} \quad \dots \quad (16)$$

$$0 = -us - z^2 + \frac{1}{1+m}\left[B'' - \tfrac{1}{2}D\Sigma(u_a s_b - s_a u_b) + \tfrac{1}{2}c_4 D^{-1}\frac{\partial}{\partial c}(Df_1')\right] \quad \dots \quad \dots \quad (17)$$

137. These are well adapted for solution by continued approximation. Equating the coefficients of ζ^c to zero in each of them, it is to be noticed that the unknowns in A′, B″, $\Sigma(u_a s_b - s_a u_b)$ have the factor m^2 at least (§ 132). If such terms be neglected in the first approximation, (16) is a simple equation to find c_4. When c_4 has been obtained $\lambda_0 + \lambda_0'$, $= 2\lambda_0$, is found from (17), since $-a(\lambda_0 + \lambda_0')$ is the coefficient of ζ^c, while the other unknowns have the factor m^2. When c_4, λ_0 have been obtained the remaining λ_i, λ'_{-i} are found by the ordinary method given in § 132. The first approximation completed, the resulting values are used for u_5, s_5 to obtain a second approximation to c_4 from (16) and to λ_0 from (17), and then to the other unknowns from § 132, and so on.

There is less disturbance of the computation sheets than would appear. The first approximation to A is found, exactly as in § 132, by omitting all unknowns. In that

to B we omit all unknowns and the term $2(1+\mathrm{m})\frac{1}{2}\mathrm{F}^{-1}\mathrm{A}$ in the coefficient of ζ^c, and then all the quantities for (16), (17) are to hand. A difference occurs in the second approximation, due to the fact that

$$\frac{\mathrm{c}_4^{(1)}}{3\mathrm{m}^2}\frac{\partial}{\partial \mathrm{c}}(D\mathrm{f}_1'), \qquad \mathrm{c}_4^{(1)}\frac{\partial \mathrm{f}_1}{\partial \mathrm{c}}$$

must be respectively included in $\mathrm{H}^{(2)}$, $\mathrm{A}^{(2)}$ (§ 131) for the unknowns other than λ_0, c_4.

138. *Small Divisors.*—The operators D^{-1}, F^{-1} introduce small divisors in the cases of long-period and monthly terms, respectively; each set of terms whose arguments differ by $2i$ involves one of the former or two of the latter. As the basis of the continued approximation was the ability to neglect terms having m^2 as a factor, approximation may become impossible, or is at least very slow, when the small divisor is of order m^2. The method used for this case is the same as that of § 29.

Taking the case of a long-period term, suppose that the corresponding coefficients are λ_0, λ_0'. The first approximation to the other λ_i, λ'_{-i} is first obtained with λ_0, λ_0' considered as unknowns, so that they are expressed as linear functions of λ_0, λ_0' and a known part. The *first* approximation to λ_0, λ_0' is then obtained by using these values of λ_i, λ'_{-i} in u_5, ϑu_5, &c., instead of neglecting them. The process thus leads to two new simultaneous linear equations for λ_0, λ_0'. When these are solved a substitution gives the first approximation to the other unknowns. In the application of this rule it was found sufficient to determine $\lambda_{\pm 1}$, $\lambda'_{\mp 1}$, in terms of λ_0, λ_0', a second approximation giving the required accuracy, and the other coefficients being determined by the ordinary method.

For the two monthly terms, corresponding to coefficients λ_0, λ_0', λ_{-1}, λ_1', the same process was followed with respect to that pair of them, say λ_0, λ_0', which had a divisor containing m^2 as a factor; the divisor of the other pair has then only m as a factor.

Section (iii.) *The Equation for z.*

139. The non-homogeneous form was retained for z, since the greater part of the series-multiplications and many of the additions had already been obtained in computing the terms of lower orders. The method for z is, therefore, the same as in previous chapters.

The known part of the expansion of $\kappa z/r^3$ (equations (16), (18), § 20) is expressed in the form

$$\frac{\kappa z}{r^3}=\kappa_1''B_4+\kappa_2''B_3+\kappa_3''B_2+\kappa_4''B_1, \text{ with } \kappa_i''=-\tfrac{3}{2}\frac{\kappa}{\rho_0{}^3}z_i.$$

Then, using the notation,

$$f\left(\frac{u}{u_0}, \frac{s}{s_0}, \frac{z}{\rho_0}\right)=f(u, s, z)\div(u_0, s_0, \rho_0),$$

for brevity of expression, we have

$$B_1 = [u_1 + s_1] \div (u_0, s_0),$$

$$B_2 = [u_2 + s_2 - \tfrac{5}{4}(u_1^2 + s_1^2) - \tfrac{3}{2}u_1 s_1 + z_1^2] \div (u_0, s_0, \rho_0),$$

$$B_3 = [u_3 + s_3 - \tfrac{5}{2}(u_1 u_2 + s_1 s_2) - \tfrac{3}{2}(u_1 s_2 + u_2 s_1) + 2z_1 z_2 + \tfrac{35}{24}(u_1^3 + s_1^3) + \tfrac{15}{8}(u_1^2 s_1 + u_1 s_1^2) - \tfrac{5}{2}z_1^2(u_1 + s_1)] \div (u_0, s_0, \rho_0),$$

$$\begin{aligned} B_4 = [u_4 + s_4 &- \tfrac{5}{4}(2u_1 u_3 + u_2^2 + 2s_1 s_3 + s_2^2) - \tfrac{3}{2}(u_1 s_3 + u_3 s_1 + u_2 s_2) + 2z_1 z_3 + z_2^2 \\ &+ \tfrac{35}{8}(u_1^2 u_2 + s_1^2 s_2) + \tfrac{15}{8}(u_1^2 s_2 + u_2 s_1^2 + 2u_1 s_1 u_2 + 2u_1 s_1 s_2) - \tfrac{5}{2}z_1^2(u_2 + s_2) - 5z_1 z_2(u_1 + s_1) \\ &- \tfrac{105}{64}(u_1^4 + s_1^4) - \tfrac{35}{16}(u_1^3 s_1 + u_1 s_1^3) - \tfrac{75}{32}u_1^2 s_1^2 + \tfrac{35}{8}z_1^2(u_1^2 + s_1^2) + \tfrac{25}{4}z_1^2 \,.\, u_1 s_1 + \tfrac{5}{4}z_1^4] \div (u_0, s_0, \rho_0). \end{aligned}$$

The terms due to Ω_1 are given by (§ 8)

$$-\tfrac{1}{2}\frac{\partial \Omega_1}{\partial z} = \mathrm{m}^2 z\{\mathfrak{b}_2 + \tfrac{3}{2}\frac{1}{a'}(u\mathfrak{c}_3 + s\bar{\mathfrak{c}}_3)\},$$

in which $\mathfrak{b}_2$, $\mathfrak{c}_3$, $\bar{\mathfrak{c}}_3$ are expanded in powers of e' and respectively multiplied by those terms in z, uz, sz which give characteristics of the fifth order.

Finally, the part of $-D^2\Sigma z_\mu$ which depends on $\mathrm{c} - \mathrm{c}_0$, $\mathrm{g} - \mathrm{g}_0$ is

$$-2\Sigma'\left(\mathrm{c}_2\frac{\partial p}{\partial \mathrm{c}} + \mathrm{g}_2\frac{\partial p}{\partial \mathrm{g}}\right)Dz_3 - \mathrm{g}_2^2 z_1 - 2\mathrm{g}_4 Dz_1',$$

with the notation for Σ' given in § 134, and for z' given in § 135. All the terms are known with the exception of g_4; this quantity is determined as in §§ 31, 73.

Section (iv). *Nature of the Computations. Tests for Accuracy.*

140. *The Computations for u_λ, s_λ.*—The equations, prepared for computation, have been given in § 131. The principal part of the labour is the formation of the products

$$u_a u_b,\ u_a s_b,\ \vartheta u_a \,.\, \vartheta s_b,\ u_a \vartheta s_b,\ Dz_a \,.\, Dz_b,\ z_a z_b, \quad (a = 4, 3;\ b = 5 - a)$$

in which $s_b = \bar{u}_b$, $\vartheta s_b = -\overline{\vartheta u_b}$.

For calculating L′, Λ′ (§ 133) we require the products

$$u_a u_b,\ u_a s_b,\ z_a z_b,\ u_a u_b u_c,\ u_a u_b s_c,$$

where $a + b \leqq 4$ in the first three and $a + b + c \leqq 4$ in the last two. Most of these, however, are deducible by briefer multiplications from the results obtained in calculating the fourth order terms for which the functions $\Sigma u_a u_b / u_0^2$, $\Sigma u_a s_b / u_0 s_0$, &c., were obtained; the multiplications of these by u_0^2, $u_0 s_0$, &c., are easy on account of the rapid convergence of all functions of u_0, s_0.

For computing $\delta\mathrm{f}$, $\delta(D\mathrm{f}')$ (§ 134) we require

$$u_a s_b,\ s_b \vartheta u_a,\ z_2 z_1,\ z_1 Dz_2,\ u_1 s_0,\ s_0 \vartheta u_1; \quad (a = 3, 2;\ b = 3 - a)$$

the first, third, and fifth of these had been previously obtained. The products in § 135 which contain an accented z are deducible by mere changes of sign from those formed without an accent.

Of the remaining calculations which do not consist of additions or subtractions there are multiplications of series by constant factors (*e.g.* G by $\frac{9}{2}\mathrm{m}^2$ in A). The operation D consists in multiplying each coefficient by the corresponding index of ζ,

while the operations D^{-1}, D^{-2}, F^{-1} consist of similar divisions. In many cases the factors are small and the divisors greater than unity, while the number of significant figures is small, so that these operations can frequently be done without the use of logarithms. It is, in fact, much easier to divide mentally by a number of, say, three figures than to multiply by such a number. CRELLE's or TAMBORREL's tables might have been used for both operations. As a matter of fact these aids were not employed. I believe that where the numbers to be multiplied consist of only three significant figures, and do not run at all consecutively, it takes less time to use a four-place table of logarithms (which can be mounted on a single card) than to be obliged to turn over the pages of a bulky volume of multiplication tables.

141. *The Computations for z.*—The processes were the same as those of previous chapters. Of the κ''_i, B_i all but κ''_4, B_4 had been previously found; κ''_4 is obtained from the brief series multiplications of z_4 by $-3\kappa/2\rho_0^3$, while all the products and most of the sums in B_4 had been obtained in computing u_4. The only multiplications remaining were the $\kappa''_i B_{5-i}$ ($i=1, 2, 3, 4$), and the uz in $\partial \Omega_1/\partial z$, most of which latter products were at hand. The long-period terms do not produce small divisions, and the initial number of places of decimals adopted was such that special methods for the monthly terms were not required. The approximations also were sufficiently rapid.

142. *Tests.*—All multiplications and additions were tested by the addition test explained in Chap. I., Sect. viii. Practically every operation comes under one of these heads. The apparent exceptions are the operations D, D^{-1}, F^{-1}. For D we have

$$D\Sigma\mu_i\zeta^{2i+\tau} = (\Sigma 2i\mu_i + \tau\Sigma\mu_i)\zeta^{2i+\tau}.$$

If we put $\zeta = 1$ we obtain a test easy to compute, since i is a small integer. The same process performed on the result tests for D^{-1} and a slight extension enables us to test similarly for F^{-1}.

This method works well for the *details* of the calculations. The calculations *en masse* can be verified by going back to the differential equations and using the above method for the operator D with $\zeta = 1$. This was not used very extensively on the fifth-order terms, for reasons which will appear in the rest of this section. Such a test is very laborious and it seemed hardly necessary to make it.

143. An indirect test in fact arises with every set of terms with a given characteristic. Each such set is u, s, and every other set in z has one (long-period) or two (monthly) terms whose coefficients possess small divisors, and the process of division by these is practically the last step in the determination of the coefficients. Now, in the fifth-order terms the chief danger of error arises not in the *details* of the calculations, which can be tested by the method just described, but in the possible omission of a whole set of terms, or in the use of a wrong set, owing to the very large number of sets to be dealt with. It is just such errors that would be detected, for even though they might not be very large the numbers which go to make up the long-period or monthly terms are always formed of the differences of numbers large in

comparison with their algebraical sum, so that an error, after the process of division by a small divisor, may cause a coefficient to appear several times its actual value; a rough comparison with DELAUNAY's or HANSEN's results would reveal the error at once. It must be remembered that such errors would practically run through the whole series of calculations, but would be mainly revealed by a few terms. Thus one of the chief causes of the extent of the calculations furnishes the most valuable test of their accuracy. Even if DELAUNAY's results were not available the test would often still work, owing to the fact that negative powers of m cannot be present, and therefore that the maximum order of magnitude of any coefficient can be roughly stated in advance.*

144. *A Special Set of Tests.*—In a paper † published in 1896 I gave some extensions of ADAMS's theorems which connected the mean motions of the perigee and node, not only with the constant term of the parallax but with one another. In fact, equations were obtained relating the parts of e, characteristics k^2, e^2k^2, k^4, e'^2k^2 respectively, with those of g, characteristics e^2, e^4, e^2k^2, $e^2e'^2$. The former set is determined in connection with u from the equations for u, s (§ 136), the latter set from the non-homogeneous equation for z. There were thus four separate tests from these relations. Further, the coefficients of certain characteristics in the constant term in the expression for the parallax are related by separate equations to parts of the motions of the perigee and node; these characteristics in the parallax are e^2, k^2, e^4, $e^2e'^2$, $k^2e'^2$, k^4, e^2k^2, furnishing seven other tests which, however, with the exception of the first two, are not very searching, owing to the small number of places of decimals used in the results for the fourth-order characteristics of the parallax.

The relations in question are included in the equations

$$\beta_2\frac{\partial\pi_1}{\partial e^2}+\beta_3\frac{\partial\theta_1}{\partial e^2}=\tfrac{3}{2}\frac{\partial}{\partial e^2}\left(\frac{E+M}{r}\right)^0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$\beta_2\frac{\partial\pi_1}{\partial k^2}+\beta_3\frac{\partial\theta_1}{\partial k^2}=\tfrac{3}{2}\frac{\partial}{\partial k^2}\left(\frac{E+M}{r}\right)^0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

where β_1, β_2, β_3 (c_1, c_2, c_3 in my previous papers) is the set of canonical constants complementary to the constants giving the position of the mean Moon, its perigee and its node at time $t=0$; π_1, θ_1 are the mean motions of the perigee and node; and $(Q)^0$ represents the mean value of a function Q.

145. I have already developed ‡ a method for the calculation of β_2, β_3 solely from the coefficients in u, s, z. The following are the results:

$$\frac{\beta_2}{na^2}=-\cdot 11844\ 44e^2-\cdot 02324e^4-\cdot 26363e^2k^2-\cdot 00110e^2e'^2,$$

$$\frac{\beta_3}{na^2}=-2\cdot 00205\ 9k^2-1\cdot 96376k^4-\cdot 28546e^2k^2-\cdot 00568e'^2k^2.$$

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* An error, the use of $u_4\zeta^{-1}$ instead of u_4/u_0 in B_4, was actually first revealed in this way, although the difference between these two series is divisible by m^2.

† "On Certain Properties of the Mean Motions," &c., *Proc. Lond. Math. Soc.* vol. 28.

‡ "On the Formation of the Derivatives of the Lunar Coordinates with Respect to the Elements," *Trans. Amer. Math. Soc.* vol. 4 (1903), pp. 234–248.

Also

$$\pi_1 = n\left(1 - \frac{c}{1+m}\right), \ \theta_1 = n\left(1 - \frac{g}{1+m}\right).$$

The values of c_0, g_0 are given in Chapter III., those of c_2, g_2 in Chapter V., those of c_4, g_4 in the following Seetion. They are

$$\begin{aligned} c = {} & +1{\cdot}07158\ 32774 + {\cdot}00268\ 571e^2 - {\cdot}03465\ 60e'^2 + {\cdot}05385\ 595k^2 - {\cdot}02212\ 6a^2 \\ & + {\cdot}00023e^4 + {\cdot}0181e^2e'^2 + {\cdot}00145e^2k^2 + {\cdot}1770e'^2k^2 + {\cdot}07657k^4, \\ g = {} & +1{\cdot}08517\ 14266 + {\cdot}00318\ 6183e^2 + {\cdot}00564\ 6535e'^2 - {\cdot}00806\ 6255k^2 + {\cdot}01110\ 58a^2 \\ & + {\cdot}00027e^4 + {\cdot}0104e^2e'^2 + 00875e^2k^2 - {\cdot}0090e'^2k^2 - {\cdot}00883k^4. \end{aligned}$$

146. Differentiate (18) with respect to k^2, and (19) with respect to e^2; the two left-hand members should then be equal. Calculate the terms with characteristics 1, e^2, k^2, e'^2 in these members; the results, which should be the same, give the following differences:

$$\left.\begin{aligned} (\beta_2)_{e^2}c_{k^2} &= -{\cdot}00637\ 894 \\ (\beta_3)_{k^2}g_{e^2} &= -{\cdot}00637\ 892 \end{aligned}\right\}\text{diff.} = {\cdot}00000\ 002;$$

$$\left.\begin{aligned} (\beta_2)_{e^2k^2}c_{e^2} + (\beta_3)_{e^2k^2}g_{e^2} + 2(\beta_3)_{k^2}g_{e^4} &= -{\cdot}00291 \\ 2(\beta_2)_{e^4}c_{k^2} + (\beta_2)_{e^2}c_{e^2k^2} &= -{\cdot}00290 \end{aligned}\right\}\text{diff.} = {\cdot}00001;$$

$$\left.\begin{aligned} (\beta_2)_{e^2k^2}c_{k^2} + (\beta_3)_{e^2k^2}g_{k^2} + 2(\beta_2)_{e^2}c_{k^4} &= -{\cdot}03252 \\ 2(\beta_3)_{k^4}g_{e^2} + (\beta_3)_{k^2}g_{e^2k^2} &= -{\cdot}03251 \end{aligned}\right\}\text{diff.} = {\cdot}00001;$$

$$\left.\begin{aligned} (\beta_2)_{e^2e'^2}c_{k^2} + (\beta_2)_{e^2}c_{e'^2k^2} &= -{\cdot}0228 \\ (\beta_3)_{e'^2k^2}g_{e^2} + (\beta_3)_{k^2}g_{e^2e'^2} &= -{\cdot}0226 \end{aligned}\right\}\text{diff.} = {\cdot}0002;$$

here $(\beta_2)_{e^2}$ denotes the coefficient of e^2 in β_2/na^2, &c.

These differences should properly be divided by 2, because $(\beta_3)_{k^2}$, which is accurately determined, is very nearly equal to 2.

The first result tests certain terms of the third order; the other three results test terms of the fifth order. The latter involve a large proportion of the *forms* for computation of the fifth-order terms and of the *results* for the terms of the fourth and lower orders.

147. Next, if we equate to zero the coefficients of 1, e^2, k^2, e'^2 in (18), (19), the coefficients of 1, e'^2 in the constant term of $1/r$ should be zero, and, since $E+M = n^2a^3$,

$$\left.\begin{aligned} \tfrac{1}{3}(\beta_2)_{e^2}c_{e^2} = -(1+m)na^3\left(\frac{1}{r}\right)^0_{e^4}, \ \tfrac{1}{3}(\beta_3)_{k^2}g_{k^2} = -(1+m)na^3\left(\frac{1}{r}\right)^0_{k^4} \\ \tfrac{2}{3}(\beta_2)_{e^2}c_{k^2} = \tfrac{2}{3}(\beta_3)_{k^2}g_{e^2} = -(1+m)na^3\left(\frac{1}{r}\right)^0_{e^2k^2} \end{aligned}\right\} \quad \dots \quad \dots \quad (20)$$

From the results of the transformation to polars (Chapter IX.) I find

$$\begin{aligned} \left(\frac{a}{r}\right)^0 = {} & +{\cdot}99999\ 971 + {\cdot}00000\ 0e^2 + {\cdot}0012\ e'^2 + {\cdot}00000\ k^2 + {\cdot}00a^2 + 0001\ e^4 - {\cdot}0001\ e^2e'^2 \\ & + {\cdot}0040\ e^2k^2 + {\cdot}000\ e'^2k^2 - {\cdot}0049\ k^4. \end{aligned}$$

It will be noticed that the differences from zero of the coefficients of e^2, k^2, $e^2e'^2$, e'^2k^2 do not exceed one unit in the last places calculated. A corresponding agreement will be found for the coefficients of e^4, e^2k^2, k^4, when calculated from (20). This degree of aceuracy should be expected, since the number of places of decimals used in trans-

forming to parallax is smaller than that actually obtained in u, z. These tests apply to the results for the fourth and lower orders.*

Section (v). *Values of* A, B, $u_\lambda \zeta^{-1}/a\lambda$; A, $\iota z_\lambda/a\lambda$.

148. The following tables show the characteristics and arguments of the terms of the fifth order which have been calculated, together with the §§ in which the results are given. I have not set forth the types of coefficients, since these are sufficiently evident from the arguments and characteristics according to the scheme adopted in Section (iv), Chapter I., and illustrated in the later chapters.

§	λ	Arguments.
149	e^5	$2i\pm5c$, $2i\pm3c$, $2i\pm c$
150	e^4e'	$2i\pm4c\pm m$, $2i\pm2c\pm m$, $2i\pm m$
151	$e^3e'^2$	$2i\pm3c\pm2m$, $2i\pm3c$, $2i\pm c\pm2m$, $2i\pm c$
152	$e^2e'^3$	$2i\pm2c\pm3m$, $2i\pm2c\pm m$, $2i\pm3m$, $2i\pm m$
153	e^3k^2	$2i\pm3c\pm2g$, $2i\pm3c$, $2i\pm c\pm2g$, $2i\pm c$
154	$e^2e'k^2$	$2i\pm2c\pm m\pm2g$, $2i\pm2c\pm m$, $2i\pm m\pm2g$, $2i\pm m$
155	ee'^2k^2	$2i\pm c\pm2m\pm2g$, $2i\pm c\pm2m$, $2i\pm c\pm2g$, $2i\pm c$
156	e'^3k^2	$2i\pm3m\pm2g$, $2i\pm3m$, $2i\pm m\pm2g$, $2i\pm m$
157	ek^4	$2i\pm c\pm4g$, $2i\pm c\pm2g$, $2i\pm c$
158	$e'k^4$	$2i\pm m\pm4g$, $2i\pm m\pm2g$, $2i\pm m$
159	e^4a	$2i_1\pm4c$, $2i_1\pm2c$, $2i_1$
160	$e^3e'a$	$2i_1\pm3c\pm m$, $2i_1\pm c\pm m$
161	e^2k^2a	$2i_1\pm2c\pm2g$, $2i_1\pm2c$, $2i_1\pm2g$, $2i_1$
162	$ee'k^2a$	$2i_1\pm c\pm m\pm2g$, $2i_1\pm c\pm m$
163	k^4a	$2i_1\pm4g$, $2i_1\pm2g$, $2i_1$
164	k^5	$2i\pm5g$, $2i\pm3g$, $2i\pm g$
165	k^3e^2	$2i\pm3g\pm2c$, $2i\pm3g$, $2i\pm g\pm2c$, $2i\pm g$
166	k^3ee'	$2i\pm3g\pm c\pm m$, $2i\pm g\pm c\pm m$
167	$k^3e'^2$	$2i\pm3g\pm2m$, $2i\pm3g$, $2i\pm g\pm2m$, $2i\pm g$
168	ke^4	$2i\pm g\pm4c$, $2i\pm g\pm2c$, $2i\pm g$
169	ke^3e'	$2i\pm g\pm3c\pm m$, $2i\pm g\pm c\pm m$
170	$ke^2e'^2$	$2i\pm g\pm2c\pm2m$, $2i\pm g\pm2c$, $2i\pm g\pm2m$, $2i\pm g$
171	kee'^3	$2i\pm g\pm c\pm3m$, $2i\pm g\pm c\pm m$
172	k^3ea	$2i_1\pm3g\pm c$, $2i_1\pm g\pm c$
173	$k^3e'a$	$2i_1\pm3g\pm m$, $2i_1\pm g\pm m$
174	ke^3a	$2i_1\pm g\pm3c$, $2i_1\pm g\pm c$
175	$ke^2e'a$	$2i_1\pm g\pm2c\pm m$, $2i_1\pm g\pm m$

* On the subject of this section see two papers by the writer: "On the Degree of Accuracy of the New Lunar Theory," &c., *Monthly Notices*, April 1904; "On the Completion of the Solution of the Main Problem in the New Lunar Theory," *ib.*, December 1904.

The terms with the following arguments and characteristics required the method of § 138 to obtain the approximations with sufficient rapidity :—

Arguments.	Characteristics.
$\pm(2-c+2m)$,	$e^3e'^2$, ee'^2k^2
$\pm(c-2g)$,	e^3k^2, ee'^2k^2, ek^4
$\pm(2+c+2m-2g)$,	ee'^2k^2
$\pm(2-c+2m-2g)$,	ee'^2k^2
± 1,	$e^4\alpha$, $e^2k^2\alpha$, $k^4\alpha$
$\pm(1-c+m)$,	$e^3e'\alpha$, $ee'k^2\alpha$
$\pm(1+2c-2g)$,	$e^2k^2\alpha$
$\pm(1+c+m-2g)$,	$ee'k^2\alpha$.

For other terms where the approximations were slow it was found to be sufficient to calculate the third approximation, since a regular law of decrease then appeared which permitted the remainders to be written down from inspection.

The results for $A/a^2\lambda$, $B/a^2\lambda$, $u_\lambda\zeta^{-1}/a\lambda$ are given in the following section under the columns headed A, B, $u_\lambda\zeta^{-1}/a\lambda$. The last is a change of plan from previous chapters, intended to increase the clearness of the reading, since the results are always referred to their arguments and not to the special notations adopted for the coefficients. The choice of subsidiary results for publication at the present time was less easy with the homogeneous equations than with the non-homogeneous. It may be stated, in this connection, that long-period small divisors do not appear in A or B except in terms which have the explicit factor m^2, while the monthly small divisors occur in B but not in A. Thus certain coefficients in the B-tables are not accurate to the last figure set down, but it is better to retain the same number of places for every coefficient under a given characteristic up to the final results. The work easily shows the extent to which the latter are correct, and in the tables for $u_\lambda\zeta^{-1}/a\lambda$ none are given beyond this point.

It is to be noted that $\overline{A}=A$, $\overline{B}=-B$, so that, given the coefficients in these functions for $\zeta^{2i+\tau}$, it is not necessary to write down those for $\zeta^{2i-\tau}$.

The coefficients of ζ^0 in B are enclosed in square brackets, to signify that they are those parts of B denoted by B″ in § 136.

In Chap. VI. the A-tables for the z-equation were given in two parts (see § 98); here the parts are not separated, since the L_λ are generally quite small and easy to compute, and, further, they were not separated on the computation sheets.

All the other arrangements of the tables are the same as those of Chap. VI.

149. *Characteristic* e^5. Values of

A

i.	$2i+5c$.	$2i+3c$.	$2i+c$.
4			+·00003
3		+·00007	+·00094
2	+·00007	+·00246	+·01912
1	+·00256	+·05054	−·00112
0	+·05446	−·00888	+·00019
−1	−·02304	−·00029	+·00021
−2	+·00038	−·00007	+·00124
−3	−·00003	+·00047	+·00371
−4	+·00004	+·00039	+·00019
Sum...	+·03444	+·04469	+·02451

B

i.	$2i+5c$.	$2i+3c$.	$2i+c$.
4			
3		−·00001	−·00014
2	−·00001	−·00035	−·00486
1	−·00036	−·01208	+·00015
0	−·01258	+·01396	[+·01315]
−1	+·00824	−·00093	+·00443
−2	−·00089	+·00051	−·00222
−3	+·00001	−·00054	−·00102
−4	−·00004	−·00011	−·00003
Sum...	−·00563	+·00045	+·00946

$u_e \zeta^{-1} \div ae^5$

i.	$2i+5c$.	$2i-5c$.	$2i+3c$.	$2i-3c$.	$2i+c$.	$2i-c$.
4				+·0001		
3				+·0002	+·0001	+·0008
2			+·0004	−·0002	+·0041	−·0004
1	+·0004	−·0004	+·0101		−·0054	−·0020
0	+·0103	+·0005	−·0124	+·0010	−·0042	−·0042
−1	−·0096		+·0030	+·0005	+·0036	+·0006
−2	+·0018		+·0001		+·0002	+·0002
−3					+·0001	
−4						
Sum...	+·0029	+·0001	+·0012	+·0016	−·0015	−·0050

$c_{e^5} = +·00023$.

150. *Characteristic* e^4e'. Values of

A

$i.$	$2i+4c+m.$	$2i+4c-m.$	$2i+2c+m.$	$2i+2c-m.$	$2i+m.$
4				+·00001	−·00004
3		+·00002	−·00018	+·00091	−·00179
2	−·00018	+·00099	−·00591	+·02674	−·02614
1	−·00734	+·02727	−·08748	+·23138	−·03696
0	−·10854	+·12325	−·07559	+·00218	−·02908
−1	−·04499	−·02582	+·00422	−·02991	+·01844
−2	−·00227	+·00166	+·00869	−·00282	+·10569
−3	+·00130	+·00025	+·01943	−·00340	+·00957
−4	+·00134	−·00016	+·00151	−·00025	+·00029
Sum...	−·16068	+·12746	−·13531	+·22484	+·03998

B

$i.$	$2i+4c+m.$	$2i+4c-m.$	$2i+2c+m.$	$2i+2c-m.$	$2i+m.$
4					
3			+·00001	−·00007	+·00027
2	+·00003	−·00009	+·00091	−·00433	+·00859
1	+·00112	−·00420	+·02694	−·08567	+·00043
0	+·03275	−·04343	+·00945	+·02907	+·00054
−1	+·01128	+·01442	−·00563	+·00055	+·02974
−2	−·00225	−·00015	−·00336	+·00121	−·04085
−3	−·00119	+·00032	−·00766	+·00130	−·00158
−4	−·00050	+·00008	−·00027	+·00003	−·00003
Sum...	+·04124	−·03305	+·02039	−·05791	−·00289

$u_{e^4e'}\zeta^{-1} \div ae^4e'$

$i.$	$2i+4c+m.$	$2i-4c-m.$	$2i+4c-m.$	$2i-4c+m.$
4		+·0004		−·0001
3		+·0016		+·0003
2		+·0007	+·0001	+·0006
1	−·0010	−·0016	+·0032	−·0009
0	−·0258	−·0016	+·0322	+·0021
−1	−·0287	−·0001	−·0216	+·0002
−2	+·0103		+·0013	
−3	+·0003		+·0001	
−4				
Sum ...	−·0449	−·0006	+·0153	+·0022

$u_{e^4e'}\zeta^{-1} \div ae^4e'$ (*continued*)

i.	$2i+2c+m.$	$2i-2c-m.$	$2i+2c-m.$	$2i-2c+m.$	$2i+m.$	$2i-m.$
4		+·0002		−·0001		+·0001
3		+·0060	+·0001	−·0010	−·0003	+·0014
2	−·0009	+·0045	+·0036	−·0014	−·0072	+·0307
1	−·0224	−·0005	+·0626	−·0245	−·0207	−·0062
0	−·0472	−·0051	−·0138	+·0008	+·0405	−·0298
−1	+·0193	−·0014	+·0294	+·0044	+·0084	−·0060
−2	+·0024		−·0005	+·0001	+·0022	−·0005
−3	+·0005		−·0001		+·0001	
−4						
Sum...	−·0483	+·0037	+·0813	−·0217	+·0230	−·0103

151. *Characteristic* $e^3e'^2$. Values of

A

i.	$2i+3c+2m.$	$2i+3c-2m.$	$2i+3c.$	$2i+c+2m.$	$2i+c-2m.$	$2i+c.$
4					+·0001	
3		+·0001			+·0055	−·0019
2	+·0002	+·0068	−·0027	−·0025	+·1347	−·0514
1	+·0030	+·1315	−·0631	−·1787	+·5806	−·2882
0	−·0484	+·1823	−·1234	−·0179	−·0163	+·0170
−1	−·0537	−·0035	−·0079	−·0638	−·0038	+·0235
−2	−·0271	+·0048	−·0041	+·2655	−·0637	−·0939
−3	+·0345	−·0063	−·0078	+·0407	−·0013	−·0130
−4	+·0038	−·0001	−·0012	+·0014		−·0004
Sum...	−·0877	+·3156	−·2102	+·0447	+·6358	−·4083

B

i.	$2i+3c+2m.$	$2i+3c-2m.$	$2i+3c.$	$2i+c+2m.$	$2i+c-2m.$	$2i+c.$
4						
3					−·0005	+·0002
2		−·0007	+·0002	−·0016	−·0181	+·0074
1	−·0012	−·0205	+·0100	−·0337	−·0616	+·0859
0	−·0210	−·0282	+·0540	+·0058	−·0887	[+·1595]
−1	−·0031	−·0235	+·0361	−·0189	−·0424	−·0473
−2	+·0120	−·0039	+·0026	−·0599	−·0123	+·0308
−3	−·0090	−·0014	+·0033	−·0053	−·0005	+·0017
−4	−·0004	−·0001		−·0002		
Sum...	−·0227	−·0783	+·1062	−·1138	−·2241	+·2382

$u_{e^3e'^2}\zeta^{-1} \div ae^3e'^2$

i.	$2i+3c+2m$.	$2i-3c-2m$.	$2i+3c-2m$.	$2i-3c+2m$.	$2i+3c$.	$2i-3c$.
3		+·016		−·002		−·004
2		+·033	+·001	−·021		+·014
1		−·013	+·016	+·015	−·007	−·020
0	−·012	−·002	+·075	+·008	−·042	−·003
−1	−·113		+·078	+·001	−·046	
−2	+·029		−·004		−·010	
−3	+·002					
Sum...	−·094	+·034	+·166	+·001	−·105	−·013

i.	$2i+c+2m$.	$2i-c-2m$.	$2i+c-2m$.	$2i-c+2m$.	$2i+c$.	$2i-c$.
3		+·006	+·001			−·002
2	−·001	+·118	+·018	−·020	−·007	−·038
1	−·051	+·042	+·235	−·041	−·101	−·068
0	−·063	−·035	+·027	+·091	−·100	−·100
−1	+·072	−·007	−·094	+·028	+·080	−·010
−2	+·014		−·003	+·001	−·004	−·001
−3						
Sum...	−·029	+·124	+·184	+·059	−·132	−·219

$c_{e^2e'^2} = +·0181.$

152. *Characteristic* $e^2e'^3$. Values of

A

i.	$2i+2c+3m$.	$2i+2c-3m$.	$2i+2c+m$.	$2i+2c-m$.	$2i+3m$.	$2i+m$.
4						
3		+·0008		−·0003		+·0001
2		+·0289	+·0015	−·0148	+ ·0064	−·0193
1	+·0010	+·4412	+·0039	−·2390	+ ·0396	+·0623
0	−·0588	+·1698	−·4221	+·2731	+ ·0994	+·3992
−1	+·6702	−·0565	−·1867	+·4332	+ ·7555	−·4246
−2	+·2282	+·0134	−·0876	−·0586	+ ·4173	−·1979
−3	+·0813	+·0014	−·0330	−·0057	+ ·0192	−·0088
−4	+·0031		−·0012		+ ·0005	−·0002
Sum...	+·9250	+·5990	−·7252	+·3879	+1·3379	−·1892

B

i.	$2i+2c+3m$.	$2i+2c-3m$.	$2i+2c+m$.	$2i+2c-m$.	$2i+3m$.	$2i+m$.
4						
3						−·0001
2		−·0017	−·0004	+·0013	+·0004	−·0030
1	−·0004	+·0242	−·0069	+·0096	+·0046	+·0178
0	−·0291	+·0075	−·1145	+·1379	+·0278	−·0457
−1	+·0431	+·0086	−·0156	−·0348	+·1365	−·0423
−2	+·1104	+·0038	−·0287	−·0223	−·0023	+·0072
−3	−·0011	+·0002	+·0010	−·0010	−·0011	+·0006
−4	−·0002		+·0001			
Sum...	+·1227	+·0426	−·1650	+·0907	+·1659	−·0655

$u_{e^2e'^3}\zeta^{-1} \div ae^2e'^3$

i.	$2i+2c+3m$.	$2i-2c-3m$.	$2i+2c-3m$.	$2i-2c+3m$.	$2i+2c+m$.	$2i-2c-m$.
3		+·012				−·005
2		+·260	+·003	+·007		−·091
1		+·138	+·059	−·017		+·064
0	−·021	−·005	+·143	+·045	−·162	−·045
−1	−·537		+·071	+·004	+·029	
−2	+·065				−·030	
−3						
Sum...	−·493	+·405	+·276	+039	−·163	−·077

i.	$2i+2c-m$.	$2i-2c+m$.	$2i+3m$.	$2i-3m$.	$2i+m$.	$2i-m$.
3		−·001		+·002		−·001
2	−·001	−·022	+·001	+·058	−·002	−·026
1	−·028	+·323	+·021	+·678	+·021	−·269
0	+·135	+·071	−·224	+·215	−1·045	+·782
−1	−·610	−·001	+·220	+·003	−·060	+·023
−2	−·003		+·003		−·001	
−3						
Sum...	−·507	+·370	+·021	+·956	−1·087	+·509

153. *Characteristic* e^3k^2. Values of

A

i.	$2i+3c+2g$.	$2i+3c-2g$.	$2i+3c$.	$2i+c+2g$.	$2i+c-2g$.	$2i+c$.
4					−·00001	−·00001
3		−·00001	−·00002	+·00004	−·00022	−·00015
2	+·00002	−·00110	−·00007	+·00024	−·01654	−·00488
1	+·00099	−·05196	+·00177	−·01714	+·00407	+·04333
0	+·00943	+·00377	+·42905	−·83981	+·00270	−·00006
−1	−·09563	−·00030	−·00666	+·02984	−·28820	+·00854
−2	+·00264	−·02607	+·00274	+·00175	−·00762	−·02586
−3	−·00015	−·00078	−·00387	−·00096	−·00002	−·00207
−4	+·00011	+·00001	−·00021	+·00006	+·00001	−·00003
Sum...	−·08259	−·07644	+·42273	−·82598	−·30583	+·01881

B

i.	$2i+3c+2g$.	$2i+3c-2g$.	$2i+3c$.	$2i+c+2g$.	$2i+c-2g$.	$2i+c$.
4					−·00001	
3		−·00001		−·00002	−·00041	−·00010
2	−·00005	−·00086	−·00014	−·00185	−·01289	−·00579
1	−·00297	−·02605	−·00869	−·07990	+·02099	−·11843
0	−·12689	−·01865	−·24098	+·04525	−·02231	[+·24621]
−1	+·06349	−·01191	+·04258	+·01589	+·01792	+·02627
−2	−·00665	+·00234	−·00183	−·00009	−·01684	−·02208
−3	−·00011	−·00121	−·00176	−·00220	−·00039	−·00131
−4	−·00012	−·00001	−·00011	−·00006		
Sum...	−·07330	−·05636	−·21093	−·02298	−·01394	+·12477

$u_{e^3k^2}\,\zeta^{-1} \div ae^3k^2$

i.	$2i+3c+2g$.	$2i-3c-2g$.	$2i+3c-2g$.	$2i-3c+2g$.	$2i+3c$.	$2i-3c$.
4		+·0001				−·0001
3		+·0003		−·0002		−·0020
2		+·0001	−·0002	−·0096		−·0064
1		−·0212	−·0164	+·0204	+·0001	+·0003
0	+·0005	+·0414	−·0332	+·0478	+·0812	+·0652
−1	−·0292	+·0008	+·0034	−·0010	−·0035	+·0030
−2	+·0104		+·0006		+·0001	
−3	−·0003		+·0004		+·0001	
−4						
Sum...	−·0186	+·0215	−·0454	+·0574	+·0780	+·0600

$u_{e^3k^2}\,\zeta^{-1} \div ae^3k^2$ (*continued*)

i.	$2i+c+2g$.	$2i-c-2g$.	$2i+c-2g$.	$2i-c+2g$.	$2i+c$.	$2i-c$.
3		−·0003	−·0001			−·0005
2		−·0029	−·0057	−·0017	−·0011	−·0151
1	−·0041	+·0040	−·0061	−·0939	−·0118	−·0280
0	−·2427	−·0019	+·0612	+·2074	−·0946	−·0946
−1	+·1319	+·0253	−·0025	−·0098	+·0570	+·0224
−2	−·0053	+·0005	+·0052	−·0005	+·0019	+·0017
−3			+·0001		+·0003	
Sum ...	−·1202	+·0247	+·0521	+·1015	−·0483	−·1141

$c_{e^3k^2} = +·00145.$

154. *Characteristic* $e^2e'k^2$. Values of

A

i.	$2i+2c+m+2g$.	$2i+2c+m-2g$.	$2i+2c-m+2g$.	$2i+2c-m-2g$.
4				+·00001
3		−·00002	+·00001	−·00005
2	−·00003	+·00092	+·00015	−·01053
1	−·00069	+·05303	+·00339	−·16500
0	+·01234	+·38403	+·02137	−·54035
−1	−·22251	−·25505	−·00741	+·01095
−2	−·06377	−·03567	+·00224	+·00588
−3	+·00236	−·00032	−·00138	+·00006
−4	+·00055	+·00003	−·00016	
Sum ...	−·27175	+·14695	+·01821	−·69903

i.	$2i+2c+m$.	$2i+2c-m$.	$2i+m+2g$.	$2i+m-2g$.	$2i+m$.
4				−·00003	+·00002
3	+·00003	−·00001	−·00002	−·00056	+·00030
2	+·00061	−·00238	+·00019	+·00092	+·01034
1	+·02468	−·02687	+·03053	+·15664	−·10067
0	+·42785	−·47278	−·01030	−·14557	−·38055
−1	+·10576	−·44694	−·81041	−·14192	−·19920
−2	−·07460	−·02770	−·03083	−·00061	−·06463
−3	−·01376	+·00113	−·00003	+·00007	−·00203
−4	−·00034	+·00003	+·00007		−·00007
Sum ...	+·47023	−·97552	−·82080	−·13106	−·73649

B

i.	$2i+2c+m+2g$.	$2i+2c+m-2g$.	$2i+2c-m+2g$.	$2i+2c-m-2g$.
4				
3		+·00003		−·00016
2	+·00005	+·00234	−·00037	−·00707
1	+·00285	+·05964	−·01526	−·11226
0	+·06395	+·00685	−·07843	+·02514
−1	+·04286	+·05958	−·02791	+·03346
−2	−·00288	−·03142	+·00113	+·00389
−3	−·00551	−·00108	+·00078	+·00015
−4	−·00029		+·00004	
Sum ...	+·10103	+·09594	−·12002	−·05685

i.	$2i+2c+m$.	$2i+2c-m$.	$2i+m+2g$.	$2i+m-2g$.	$2i+m$.
4					
3	+·00001	−·00001	+·00001	+·00078	+·00039
2	+·00096	−·00282	+·00132	+·01530	+·01883
1	+·05111	−·12626	+·03039	+·00096	−·01694
0	+·31953	−·31454	−·10006	+·16933	+·02876
−1	−·01784	+·01876	+·04470	−·16277	−·20480
−2	−·06593	−·00962	−·06061	−·00835	−·05815
−3	−·00757	+·00186	−·00299	−·00016	−·00128
−4	−·00015	+·00002	−·00006		
Sum ...	+·28012	−·43261	−·08730	+·01509	−·23319

$u_{e^2e'k^2}\,\zeta^{-1} \div ae^2e'k^2$

i.	$2i+2c+m+2g$.	$2i-2c-m-2g$.	$2i+2c+m-2g$.	$2i-2c-m+2g$.	$2i+2c-m+2g$.	$2i-2c+m-2g$.
4		+·0001				
3		+·0035		−·0001		−·0015
2		+·0043	+·0001	−·0055		−·0001
1		−·0840	+·0258	−·1856	+·0002	−·0077
0	+·0001	−·0489	+·1474	−·2308	+·0011	+·0676
−1	−·0816	−·0013	−·0327	−·0014	−·0008	+·0057
−2	+·0500		+·0229	−·0003	−·0049	+·0001
−3	+·0007		+·0005		−·0003	
−4	+·0001					
Sum ...	−·0307	−·1263	+·1640	−·4237	−·0047	+·0641

$u_{e^2e'k^2}\,\zeta^{-1} \div ae^2e'k^2$ (*continued*)

i.	$2i+2c-m-2g$.	$2i-2c+m+2g$.	$2i+2c+m$.	$2i-2c-m$.	$2i+2c-m$.	$2i-2c+m$.
4						
3				−·0027		+·0002
2	−·0015	+·0007	+·0001	−·0851	−·0004	−·0126
1	−·1024	+·0154	+·0060	−·0627	−·0114	−·3298
0	−·1113	+·2490	+·2517	−·0455	−·3504	+·0044
−1	+·0015	−·0190	−·0154	−·0102	+·6405	+·0216
−2	−·0027	+·0005	+·0068	−·0002	−·0052	+·0005
−3	−·0001		+·0022		−·0005	
−4			+·0001			
Sum ...	−·2165	+·2466	+·2515	−·2064	+·2726	−·3157

i.	$2i+m+2g$.	$2i-m-2g$.	$2i+m-2g$.	$2i-m+2g$.	$2i+m$.	$2i-m$.
4						
3			−·0001		+·0001	−·0003
2	+·0001	−·0240	−·0018	−·0003	+·0019	−·0127
1	+·0032	−·0536	−·0170	−·0174	−·0014	−·2568
0	+·0312	−·0490	−·0137	−·1218	+·7122	−·5019
−1	+·4615	−·0236	+·1196	−·0690	+·1027	−·0550
−2	−·0093	−·0006	+·0034	+·0008	+·0142	−·0044
−3	+·0003			−·0001	+·0003	−·0001
−4						
Sum ...	+·4870	−·1508	+·0904	−·2078	+·8300	−·8312

155. *Characteristic* ee'^2k^2. Values of

A

i.	$2i+c+2m+2g$.	$2i+c+2m-2g$.	$2i+c-2m+2g$.	$2i+c-2m-2g$.
3			+·0001	+·0005
2	−·0001	−·0008	+·0005	−·0179
1	−·0002	+·0033	+·0105	+·1473
0	+·0390	−·0123	+·0582	−·1332
−1	−·3226	−·2290	+·0110	−·0001
−2	−·0546	−·0024	+·0030	−·0009
−3	+·0144	+·0003	−·0001	
Sum ...	−·3241	−·2409	+·0832	−·0043

i.	2*i*+c+2m.	2*i*+c−2m.	2*i*+c+2g.	2*i*+c−2g.	2*i*+c.
3		−·0003		−·0004	+·0002
2	−·0007	−·0181	−·0004	−·0046	+·0078
1	−·0202	−·3772	+·0003	−·0352	+·2580
0	+·0625	+·3606	+·0137	+·0671	+·1464
−1	+·2677	−·0326	+·0364	+·0867	+·1292
−2	−·2448	+·0080	+·0312	+·0018	+·0505
−3	−·0078	−·0002	−·0080		+·0021
Sum ...	+·0567	−·0598	+·0732	+·1154	+·5942

B

i.	2*i*+c+2m+2g.	2*i*+c+2m−2g.	2*i*+c−2m+2g.	2*i*+c−2m−2g.
3				−·0007
2		−·0009	− ·0010	−·0204
1	−·0004	−·0817	− ·0272	−·6451
0	−·0165	−·0493	+ ·1033	−·2671
−1	−·3784	+·0744	+1·3216	−·0042
−2	−·0444	−·0089	+ ·0107	−·0002
−3	−·0052	−·0001	− ·0003	
Sum ...	−·4449	−·0665	+1·4071	−·9377

i.	2*i*+c+2m.	2*i*+c−2m.	2*i*+c+2g.	2*i*+c−2g.	2*i*+c.
3				+ ·0002	
2	−·0002	− ·0137		+ ·0125	+·0077
1	+·0172	− ·4509	+·0081	+ ·3700	+·2878
0	+·2883	−1·2136	−·0228	+1·4800	[−·0705]
−1	−·9542	− ·8189	−·0058	− ·0284	−·4794
−2	−·1850	+ ·0053	+·0070	+ ·0020	+·0924
−3	−·0039	− ·0001	+·0029		+·0019
Sum ...	−·8378	−2·4919	−·0106	+1·8363	−·1601

$$u_{e'^2k^2}\zeta^{-1} \div aee'^2k^2$$

i.	$2i+c+2m+2g$.	$2i-c-2m-2g$.	$2i+c+2m-2g$.	$2i-c-2m+2g$.	$2i+c-2m+2g$.	$2i-c+2m-2g$.
3		+·002				
2		+·059		−·001		+·014
1		−·242	−·057	−·030	+·001	−·32
0		−·037	−·031	+·036	+·004	+·087
−1	−·240		+·333	+·038	+·27	+·022
−2	+·157		+·010		−·090	
−3	+·002					
Sum ...	−·081	−·218	+·255	+·043	+·185	−·197

i.	$2i+c-m-2g$.	$2i-c+m+2g$.	$2i+c+2m$.	$2i-c-2m$.	$2i+c-2m$.	$2i-c+2m$.
3				− ·001		
2	−·002			− ·043	− ·002	+ ·003
1	−·361	− ·001	− ·001	− ·776	− ·073	+ ·341
0	+·070	− ·049	+ ·161	− ·515	− ·898	+1·433
−1	−·098	+1·249	+1·006	− ·010	− ·892	+ ·116
−2	−·001	+ ·007	+ ·072		− ·008	+ ·003
−3			+ ·001			
Sum ...	−·392	+1·206	+1·239	−1·345	−1·873	+1·896

i.	$2i+c+2g$.	$2i-c-2g$.	$2i+c-2g$.	$2i-c+2g$.	$2i+c$.	$2i-c$.
3		−·002				
2		−·061	− ·001		+ ·001	+·008
1		+·098	+ ·167	+·009	+ ·038	−·164
0		+·006	−1·280	+·421	+ ·199	+·199
−1	+·086	−·007	− ·060	−·827	+1·038	−·071
−2	−·038		− ·002	−·004	− ·030	−·001
−3	−·001				− ·001	
Sum ...	+·047	+·034	−1·176	−·401	+1·245	−·029

$c_{e'^2k^2} = +·1770$.

156. *Characteristic* e'^3k^2. Values of

A

i.	$2i+3m+2g$.	$2i+3m-2g$.	$2i+3m$.	$2i+m+2g$.	$2i+m-2g$.	$2i+m$.
3					+·0004	−·0001
2		+·0018	+ ·0002	+·0002	+·0025	−·0027
1	+·0004	+·0349	− ·0080	+·0017	−·0436	−·0190
0	+·1256	+·1251	− ·2031	+·0174	+·0292	+·1769
−1	+·3169	+·0186	− ·9697	−·1714	·0000	+·6411
−2	+·0920	+·0011	− ·0504	−·0850	−·0004	+·0253
−3	+·0047	+·0001	− ·0011	−·0029		+·0005
Sum...	+·5396	+·1816	−1·2321	−·2400	−·0119	+·8220

B

i.	$2i+3m+2g$.	$2i+3m-2g$.	$2i+3m$.	$2i+m+2g$.	$2i+m-2g$.	$2i+m$.
3						
2		−·0021	− ·0001	+·0001	−·0037	−·0013
1	−·0002	−·0187	− ·0184	−·0003	+·0201	−·0116
0	−·0698	+·1567	+ ·1648	−·1380	+·1626	−·1038
−1	−·2423	+·0103	−1·5943	+·1286	−·0057	+·8841
−2	−·0216	−·0006	− ·0320	+·0319		+·0210
−3	−·0008		− ·0004	+·0008		+·0002
Sum...	−·3347	+·1456	−1·4804	+·0231	+·1733	+·7886

$u_{e'^3k^2}\zeta^{-1} \div ae'^3k^2$

i.	$2i+3m+2g$.	$2i-3m-2g$.	$2i+3m-2g$.	$2i-3m+2g$.	$2i+3m$.	$2i-3m$.
3		+·001				
2		+·021				−·006
1		+·258	+·036	+·001		−·203
0	+·002	+·027	−·117	+·009	+ ·204	−·388
−1	−·637		+·054	−·019	+1·289	+·003
−2	+·034		+·001	+·002	+ ·010	
−3						
Sum...	−·601	+·307	−·026	−·007	+1·503	−·594

$u_{e'^3k^2}\zeta^{-1} \div ae'^3k^2$ (*continued*)

i.	2*i*+m+2g.	2*i*−m−2g.	2*i*+m−2g.	2*i*−m+2g.	2*i*+m.	2*i*−m.
3						
2		−·016				+·003
1		−·380	−·016		−·002	+·097
0		+·110	−·150	−·001	+·091	+·014
−1	+·586		−·018	+·027	−·688	+·003
−2	−·031			+·003	−·006	
−3						
Sum...	+·555	−·286	−·184	+·029	−·605	+·117

157. *Characteristic* ek^4. Values of

A

i.	2*i*+c+4g.	2*i*+c−4g.	2*i*+c+2g.	2*i*+c−2g.	2*i*+c.
3		−·00008	+·00001	−·00003	−·00002
2		+·00093	+·00005	+·00131	+·00007
1	+·00002	+·02617	+·00443	−·02682	−·01225
0	−·00024	−·04835	+·03463	−·26771	+·05994
−1	+·00379	−·00499	−·00095	+·01416	+·02155
−2	+·00061	−·00007	+·00313	−·00033	+·00218
−3	−·00013		−·00075	−·00002	+·00013
Sum...	+·00405	−·02639	+·04055	−·27944	+·07160

B

i.	2*i*+c+4g.	2*i*+c−4g.	2*i*+c+2g.	2*i*+c−2g.	2*i*+c.
3		−·00010			− ·00001
2		+·00177	−·00001	− ·00011	− ·00058
1	−·00004	+·09268	−·00427	+ ·18160	− ·04036
0	−·00213	+·00898	−·80476	−3·15694	[−1·04073]
−1	+·04491	−·00200	+·36829	− ·16987	+ ·29027
−2	−·00518	−·00002	+·00192	− ·00136	− ·00736
−3	−·00017		−·00036	− ·00001	− ·00008
Sum...	+·03739	+·10131	−·43919	−3·14669	− ·79885

$u_{ek^4}\zeta^{-1} \div aek^4$

i.	2*i*+c+4g.	2*i*−c−4g.	2*i*+c−4g.	2*i*−c+4g.
3		+·0002		
2		+·0009	−·0028	
1		−·0604	+·0207	−·0001
0		+·0014	−·4579	−·0027
−1			−·0024	+·0959
−2	+·0082			−·0077
−3	−·0002			
Sum ...	+·0080	−·0579	−·4424	+·0854

i.	2*i*+c+2g.	2*i*−c−2g.	2*i*+c−2g.	2*i*−c+2g.	2*i*+c.	2*i*−c.
3		−·0002				
2		−·0074	− ·0011			+·0004
1		−·0034	− ·2171	− ·0074	+·0004	−·0805
0	+·0013	+·4813	+4·5387	−1·9841	+·3342	+·3342
−1	+·0032	+·0002	− ·0484	+ ·7232	+·1008	−·0109
−2	−·0089		− ·0008	− ·0017	−·0008	−·0001
−3	+·0001					
Sum...	−·0043	+·4705	+4·2713	−1·2700	+·4346	+·2431

$c_{k^4} = +{\cdot}07657.$

158. *Characteristic* $e'k^4$. Values of

A

i.	2*i*+m+4g.	2*i*+m−4g.	2*i*+m+2g.	2*i*+m−2g.	2*i*+m.
3		−·00015	−·00001	+·00006	−·00002
2		+·00730	+·00006	+·00526	−·00076
1	+·00002	−·01212	+·00133	−·04200	−·00520
0	−·00025	+·00155	+·03301	+·00136	−·05614
−1	+·04376	+·00005	+·02084	−·00414	−·05545
−2	−·00811		−·01413	−·00015	+·00265
−3	+·00034		−·00039		+·00003
Sum...	+·03576	−·00337	+·04071	−·03961	−·11489

B

i.	$2i+m+4g$.	$2i+m-4g$.	$2i+m+2g$.	$2i+m-2g$.	$2i+m$.
3		+·00019		+·00002	−·00001
2		−·00464	+·00008	+·00316	−·00139
1		+·05167	+·00825	+·00155	−·09713
0	−·00157	+·00266	+·45195	−·50959	−·03209
−1	−·09909	−·00005	+·08329	−·03351	+·22527
−2	+·00821		−·01241	−·00034	+·00230
−3	−·00042		−·00013	−·00001	+·00001
Sum...	−·09287	+·04983	+·53103	−·53872	+·09696

$u_{e'k^4}\zeta^{-1} \div ae'k^4$

i.	$2i+m+4g$.	$2i-m-4g$.	$2i+m-4g$.	$2i-m+4g$.
3		+·0002	−·0001	
2		+·0030	+·0013	
1		−·1301	+·0771	
0		+·0020	+·0026	
−1	+·0007		+·0001	−·0006
−2	+·0198			−·0154
−3	+·0003			−·0002
Sum ...	+·0208	−·1249	+·0810	−·0162

i.	$2i+m+2g$.	$2i-m-2g$.	$2i+m-2g$.	$2i-m+2g$.	$2i+m$.	$2i-m$.
3		−·0001				
2		−·0040	+·0016		−·0001	+·0003
1		−·1026	+·1115	−·0002	−·0044	+·0036
0	+·0011	+·1848	−·1833	−·0024	−·1037	+·0735
−1	+·0472	+·0022	−·0068	−·0650	+·1338	−·0387
−2	+·0133		−·0001	−·0064	+·0002	
−3	+·0001					
Sum...	+·0617	+·0803	−·0771	−·0740	+·0258	+·0387

159. *Characteristic* $e^4\alpha$. Values of

A

$2i$.	$2i+4c$.	$2i+2c$.	$2i$.
7		+·0001	+·0001
5	+·0001	+·0013	−·0314
3	+·0022	−·0794	−·1150
1	−·0670	−·0931	+·0027
−1	+·1415	+·0106	
−3	+·0041	+·0098	
−5	+·0025	−·0320	
−7	−·0029	−·0054	
Sum	+·0805	−·1881	−·1436

B

$2i$.	$2i+4c$.	$2i+2c$.	$2i$.
7			+·0001
5		+·0001	+·0045
3	+·0002	+·0138	−·0001
1	+·0152	−·0110	+·0238
−1	−·0277	−·0171	
−3	+·0008	+·0002	
−5	−·0012	+·0004	
−7	+·0001	+·0004	
Sum	−·0126	−·0132	+·0283

$u_{e^4\alpha}\zeta^{-1} \div ae^4\alpha$

$2i$.	$2i+4c$.	$2i-4c$.	$2i+2c$.	$2i-2c$.	$2i$.
7		−·001		−·001	
5		−·004		−·013	−·005
3		+·002	−·013	−·013	−·041
1	−·012	+·003	−·023	+·036	+·020
−1	+·051		+·087	−·005	−·095
−3	+·008		−·008	−·001	−·006
−5	−·003		−·002		
−7					
Sum ...	+·044	·000	+·041	+·003	−·127

160. *Characteristic* $e^3e'\alpha$. Values of

A

$2i$.	$2i+3c+m$.	$2i+3c-m$.	$2i+c+m$.	$2i+c-m$.
7				+ ·0009
5	−·0001	+ ·0015	+ ·0110	+ ·0170
3	+·0161	+ ·0275	+ ·5502	− ·4322
1	+·7904	− ·1771	+2·5540	− ·1684
−1	−·0381	−1·3735	+ ·5578	+1·2147
−3	−·2193	− ·5492	− ·2981	+ ·3211
−5	−·0708	− ·0196	− ·1847	+ ·1151
−7	−·0196	+ ·0083	+ ·0034	+ ·0029
Sum ...	+·4586	−2·0821	+3·1936	+1·0711

B

$2i$.	$2i+3c+m$.	$2i+3c-m$.	$2i+c+m$.	$2i+c-m$.
7				
5			−·0031	+·0029
3	−·0059	+·0034	−·1368	+·0341
1	−·2621	+·0198	+·8794	−·0464
−1	+·1203	−·3479	−·0729	−·1214
−3	−·0134	+·0400	−·0230	+·1002
−5	−·0220	−·0180	+·0124	−·0288
−7	+·0013	−·0022	+·0006	−·0007
Sum ...	−·1818	−·3049	+·6566	−·0601

$u_{e^3e'a}\zeta^{-1} \div ae^3e'a$

$2i$.	$2i+3c+m$.	$2i-3c-m$.	$2i+3c-m$.	$2i-3c+m$.	$2i+c+m$.	$2i-c-m$.	$2i+c-m$.	$2i-c+m$.
7		−·004		+·002		−·001		+·001
5		−·065		−·011	+·003	−·032	−·002	+·027
3	+·005	−·056	−·001	−·116	+·130	−·238	−·073	+·179
1	+·183	+·003	−·039	−·190	+1·064	−·106	−·171	+1·95
−1	−·055	+·013	−·668	−·004	−·286	+·374	−2·41	−·030
−3	+·150		+·457	−·001	−·072	+·009	+·033	−·004
−5	−·014		−·004		−·002		+·002	−·001
−7	−·001							
Sum	+·268	−·109	−·255	−·320	+·837	+·006	−2·621	+2·122

161. *Characteristic* e^2k^2a. Values of

A

$2i$.	$2i+2c+2g$.	$2i+2c-2g$.	$2i+2c$.	$2i+2g$.	$2i$.
7					+·0001
5		−·0043		−·0007	+·0026
3	−·0001	−·0272	+·0271	−·0243	+·5215
1	+·0304	+·0033	+1·0340	+·4413	−·1873
−1	+·3747	+·0179	−·1647	+·0615	
−3	+·0540	+·1558	−·1047	−·0183	
−5	+·0111	−·0058	+·0813	−·0200	
−7	−·0040	−·0002	−·0006	−·0011	
Sum ...	+·4661	+·1395	+·8724	+·4384	+·3369

B

2i.	2i+2c+2g.	2i+2c−2g.	2i+2c.	2i+2g.	2i.
7					+·0001
5		+·0002		+·0003	+·0067
3	+·0013	+·0212	+·0138	+·0435	+·2328
1	+·0956	+·0894	+·4010	−·1350	+·7293
−1	+·0410	−·0715	−·5640	+·2047	
−3	+·0415	−·0320	+·1863	+·1826	
−5	+·0049	+·0049	+·0339	+·0079	
−7	+·0006	−·0001	+·0005		
Sum...	+·1849	+·0121	+·0715	+·3040	+·9689

$u_{e^2k^2a}\zeta^{-1} \div ae^2k^2a$

2i.	2i+2c+2g.	2i−2c−2g.	2i+2c−2g.	2i−2c+2g.	2i+2c.	2i−2c.	2i+2g.	2i−2g.	2i.
7		−·002							
5		−·014				+·027		−·011	+ ·003
3		+·027	−·021	+·033	+ ·007	+·269	+·002	+·066	+ ·189
1	+·002	+·062	−·069	−·053	+ ·335	+·501	+·151	−·166	+ ·948
−1	+·111	−·030	+·162	−·150	−1·251	−·021	+·010	−·061	−1·693
−3	+·050		−·048	−·005	− ·054	−·003	−·362	−·016	− ·026
−5	−·024		−·002		− ·007		−·003		− ·001
−7	−·001								
Sum...	+·138	+·043	+·022	−·175	− ·970	+·773	−·202	−·188	− ·580

162. *Characteristic* $ee'k^2a$. Values of

A

2i.	2i+c+m+2g.	2i+c+m−2g.	2i+c−m+2g.	2i+c−m−2g.	2i+c+m.	2i+c−m.
7				− ·0003		− ·0001
5	+·0002	+ ·0070	+·0001	− ·0333	− ·0010	− ·0019
3	+·0065	+ ·0457	+·0055	− ·0382	− ·1374	+ ·0678
1	+·0766	−2·3608	+·1939	+ ·1489	−5·9398	+1·3612
−1	−·0163	− ·2389	−·7781	−1·7625	− ·2096	−4·0717
−3	−·3091	− ·0729	+·0376	+ ·0011	+1·2745	−1·4012
−5	−·1008	− ·0020	+·0530	+ ·0014	− ·0282	− ·0388
−7	−·0028		+·0009	+ ·0001	− ·0009	− ·0002
Sum...	−·3457	−2·6219	−·4871	−1·6828	−5·0424	−4·0849

B

$2i$.	$2i+c+m+2g$.	$2i+c+m-2g$.	$2i+c-m+2g$.	$2i+c-m-2g$.	$2i+c+m$.	$2i+c-m$.
7				−·0001		
5		−·0031		+ ·0009	− ·0010	+·0010
3	−·0083	−·1471	+ ·0045	+ ·1262	− ·0943	+·0747
1	−·5658	−·1190	+ ·0279	+ ·0681	−2·9362	+·9469
−1	+·4864	−·3057	−1·5007	+1·8829	− ·1327	−·3019
−3	+·0646	+·0295	− ·1013	− ·1163	+1·1756	−·8338
−5	−·0059	+·0011	− ·0178	− ·0017	+ ·0249	−·0238
−7	+·0005		− ·0003		+ ·0004	−·0001
Sum...	−·0285	−·5443	−1·5877	+1·9600	−1·9633	−·1370

$u_{ee'k^2a}\zeta^{-1} \div aee'k^2a$

$2i$.	$2i+c+m+2g$.	$2i-c-m-2g$.	$2i+c+m-2g$.	$2i-c-m+2g$.	$2i+c-m+2g$.	$2i-c+m-2g$.
7		−·001				+ ·001
5		−·088	+·002			+ ·037
3		−·320	+·074	+ ·002		+ ·212
1	+·004	−·142	−·21	+ ·014	+·005	+1·054
−1	−·012	+·392	+·268	+1·33	−·303	− ·029
−3	+·622	+·004	−·047	+ ·056	−·193	− ·001
−5	−·051		−·001	+ ·001	+·016	
−7	−·001					
Sum...	+·562	−·155	+·086	+1·403	−·475	+1·274

$2i$.	$2i+c-m-2g$.	$2i-c+m+2g$.	$2i+c+m$.	$2i-c-m$.	$2i+c-m$.	$2i-c+m$.
7						
5	− ·005		− ·001	+ ·013	+ ·001	− ·009
3	− ·211	− ·005	− ·048	+ ·611	+ ·034	− ·857
1	−1·040	− ·687	−3·090	+1·549	+ ·850	−5·45
−1	−1·315	+1·137	− ·698	− ·482	+7·75	− ·326
−3	+ ·084	− ·249	− ·536	+ ·027	+ ·005	− ·014
−5	+ ·001	− ·001	− ·005		+ ·007	
−7						
Sum...	−2·486	+ ·195	−4·378	+1·718	+8·647	−6·656

163. *Characteristic* $k^4 a$. Values of

A

$2i.$	$2i+4g.$	$2i+2g.$	$2i.$
7			
5		+·0002	−·0003
3		+·0064	−·0497
1	+·0029	+·0228	+·2695
−1	+·0658	−·2685	
−3	+·0425	−·0693	
−5	+·0025	−·0001	
−7	−·0002		
Sum ...	+·1135	−·3085	+·2195

B

$2i.$	$2i+4g.$	$2i+2g.$	$2i.$
7			
5		+ ·0001	− ·0003
3		+ ·0025	− ·0051
1	+·0013	+ ·2066	−4·7553
−1	−·0176	−2·6677	
−3	+·0765	+ ·2237	
−5	+·0043	+ ·0006	
−7		− ·0001	
Sum ...	+·0645	−2·2343	−4·7607

$u_{\mathrm{k}^4 a}\zeta^{-1} \div a k^4 a$

$2i.$	$2i+4g.$	$2i-4g.$	$2i+2g.$	$2i-2g.$	$2i.$
7					
5		−·005		+ ·002	
3		+·049		+ ·139	− ·022
1		+·190	−·006	+ ·728	−2·371
−1	+·002	+·001	−·921	+ ·715	+5·931
−3	−·005		−·066	+ ·004	− ·015
−5	−·010		−·004		
−7					
Sum ...	−·013	+·235	−·997	+1·588	+3·523

164. *Characteristic* k^5. Values of

	A		*B*
i.	$2i+5g$.	$2i+3g$.	$2i+g$.
3	−·00004	+·00004	+·00002
2	+·00002	−·00003	+·00027
1	·00000	−·00056	−·01230
0	+·00036	+·00672	−·01946
−1	+·00345	−·01322	−·00932
−2	+·00221	+·00029	+·00224
−3	+·00001	−·00001	+·00009
Sum ...	+·00601	−·00677	−·03846

$\sqrt{-1}\,z_{k^5} \div ak^5$

i.	$2i+5g$.	$2i+3g$.	$2i+g$.
3			
2			+·00001
1		−·00002	−·00147
0	+·00001	+·00067	0
−1	+·00033	−·03317	+·02556
−2	+·00258	+·00021	+·00035
−3	−·00005		
Sum ...	+·00287	−·03231	+·02445

$g_{k^5} = -·00883$

165. *Characteristic* k^3e^2. Values of

	A					*B*
i.	$2i+3g+2e$.	$2i+3g-2e$.	$2i+3g$.	$2i+g+2e$.	$2i+g-2e$.	$2i+g$.
4		−·00002			−·00013	−·00001
3		−·00038	− ·00001	− ·00002	−·00554	−·00098
2	+·00002	−·01872	− ·00156	− ·00102	−·10446	−·02841
1	+·00132	−·61256	− ·09289	+ ·00081	+·00561	−·23460
0	+·02515	+·00677	−3·34807	+1·66897	+·03331	+·01907
−1	−·45559	−·00667	+ ·04340	− ·00167	+·16980	−·00656
−2	+·00526	+·00524	− ·00258	− ·00123	+·00455	+·03469
−3	−·00002	+·00007	+ ·00064	+ ·00041	+·00012	+·00074
−4	−·00006		− ·00002	− ·00004		+·00003
Sum ...	−·42392	−·62627	−3·40109	+1·66621	+·10326	−·21603

$\sqrt{-1}z_{k^3e^2} \div ak^3e^2$

i.	2*i*+3g+2c.	2*i*+3g−2c.	2*i*+3g.	2*i*+g+2c.	2*i*+g−2c.	2*i*+g.
4						
3		−·00001			−·00025	−·00002
2		−·00079	−·00003	−·00002	−·01397	−·00117
1	+·00003	−·07184	−·00368	+·00012	+·00619	−·02814
0	+·00088	+·1018	−·35528	+·18055	−·5728	0
−1	−·04391	+·01382	+·09779	+·00192	+·01989	+·02079
−2	+·00605	+·00076	+·00223	+·00204	+·00019	+·00478
−3	−·00006		+·00011	+·00007		+·00004
−4	−·00001					
Sum	−·03702	+·0437	−·25886	+·18468	−·5607	−·00372

$g_{k^3e^2} = +\cdot 00875$

166. *Characteristic* k³ee′. Values of

A

i.	2*i*+3g+c+m.	2*i*+3g−c−m.	2*i*+3g+c−m.	2*i*+3g−c+m.
4				
3		+·0004		
2		−·0038	+·0001	+·0004
1	−·0001	−·1546	+·0058	+·0232
0	+·0190	+·2873	+·0360	−·4097
−1	−·6399	+·1250	+·1364	−·3864
−2	−·0138	−·0023	−·0023	+·0074
−3	−·0015	+·0003	+·0003	−·0002
−4	+·0003			
Sum ...	−·6360	+·2523	+·1763	−·7653

i.	2*i*+g+c+m.	2*i*+g−c−m.	2*i*+g+c−m.	2*i*+g−c+m.
4		−·0002		
3	+ ·0002	−·0050	− ·0004	−·0002
2	+ ·0026	−·1081	− ·0099	+·0096
1	+ ·0793	−·5617	− ·1619	−·3477
0	+1·3350	+·4505	−1·6058	−·7741
−1	− ·1471	−·0756	− ·4687	+·3263
−2	+ ·0033	−·0012	+ ·0173	+·0129
−3	− ·0012	+·0001	+ ·0011	+·0002
−4	− ·0007			
Sum ...	+1·2714	−·3012	−2·2283	−·7730

$\sqrt{-1}z_{k'ee'} \div ak^3ee'$

i.	$2i+3g+c+m$.	$2i+3g-c-m$.	$2i+3g+c-m$.	$2i+3g-c+m$.
3				
2		−·0001		
1		−·0098	+·0002	+·0013
0	+·0009	+·0880	+·0022	−·1026
−1	−·1385	−·1080	+·0353	+·3498
−2	+·0153	−·0015	+·0017	+·0064
−3	−·0010		+·0002	
Sum ...	−·1233	−·0314	+·0396	+·2549

i.	$2i+g+c+m$.	$2i+g-c-m$.	$2i+g+c-m$.	$2i+g-c+m$.
3		−·0002		
2	+·0001	−·0078	−·0003	+·0005
1	+·0050	−·2215	−·0109	−·1057
0	+·3491	−·3813	−·5112	+·6618
−1	+·1272	−·0260	+·4053	+·1364
−2	+·0025	−·0001	+·0088	+·0010
−3	−·0001		+·0001	
Sum ...	+·4838	−·6369	−·1082	+·6940

167. *Characteristic* $k^3e'^2$. Values of

	A					*B*
i.	$2i+3g+2m$.	$2i+3g-2m$.	$2i+3g$.	$2i+g+2m$.	$2i+g-2m$.	$2i+g$.
3		−·0002			−·0005	+·0003
2	−·0004	+·0003	−·0001	−·0001	−·0242	+·0091
1	+·0002	+·0082	+·0004	−·0023	−·3466	+·1547
0	+·0244	+·0374	+·0082	+·0646	+·0539	−·0195
−1	−·2550	+·0022	+·0993	−·0734	·0000	+·0083
−2	+·0279	−·0014	−·0180	−·0109	+·0008	+·0066
−3	−·0011	−·0002	+·0008	−·0010		+·0004
Sum ...	−·2040	+·0463	+·0906	−·0231	−·3166	+·1599

$\sqrt{-1}z_{k^3e'^2} \div ak^3e'^2$.

i.	$2i+3g+2m$.	$2i+3g-2m$.	$2i+3g$.	$2i+g+2m$.	$2i+g-2m$.	$2i+g$.
3						
2					−·0011	+·0004
1		+·0003			−·0473	+·0186
0	+·0020	+·0047	+·0012	+·175	−·168	0
−1	−·3075	+·13	+·250	+·117	+·11	−·026
−2	−·0286	−·0006	+·024	−·0015	+·0003	+·0009
−3	−·0003		+·0002			
Sum...	−·3344	+·13	+·275	+·290	−·11	−·006

$g_{k^3e'^2} = -·0090$

168. *Characteristic* ke^4. Values of

	A				*B*
i.	$2i+g+4c$.	$2i+g-4c$.	$2i+g+2c$.	$2i+g-2c$.	$2i+g$.
4		+·00086		+·00089	+·00018
3	+·00001	+·00139	+·00066	+·01631	+·00642
2	+·00084	−·00018	+·02200	+·00715	+·12224
1	+·02860	+·00114	+·42113	−·00042	−·00718
0	+·55401	−·07830	−·06698	+·00067	+·00061
−1	−·14499	−·01869	−·00019	−·03848	+·00043
−2	+·00098	−·00088	+·00010	−·01347	−·00719
−3	+·00003	−·00003	−·00083	−·00065	−·00376
−4	−·00006		−·00045	−·00002	−·00018
Sum...	+·43942	−·09469	+·37544	−·02802	+·11157

$\sqrt{-1}z_{ke^4} \div ake^4$

i.	$2i+g+4c$.	$2i+g-4c$.	$2i+g+2c$.	$2i+g-2c$.	$2i+g$.
4		+·00004		+·00002	
3		+·00021	+·00001	+·00070	+·00013
2	+·00001	+·00024	+·00044	+·00096	+·00495
1	+·00054	+·00389	+·01610	+·00193	−·00085
0	+·02001	−·00863	−·00723	−·0121	0
−1	−·01421	−·00073	−·00085	−·00473	−·00117
−2	+·00114	−·00002	−·00015	−·00055	−·00099
−3	−·00005		−·00013	−·00001	−·00016
−4	−·00001		−·00002		
Sum...	+·00743	−·00500	+·00817	−·0138	+·00191

$g_{e^4} = +·00027$

169. *Characteristic* ke^3e'. Values of

A

i.	$2i+g+3c+m$.	$2i+g-3c-m$.	$2i+g+3c-m$.	$2i+g-3c+m$.
4		+·0027		−·0002
3		+·0418	+·0002	−·0057
2	−·0018	+·0191	+·0074	+·0059
1	−·0464	−·0314	+·1638	−·0061
0	−·6708	+·0736	+·8067	−·0580
−1	−·2353	+·0479	−·0775	−·0865
−2	+·0005	+·0023	+·0025	−·0071
−3	−·0014		0000	−·0002
−4	−·0015		+·0003	
Sum ...	−·9567	+·1560	+·9034	−·1579

i.	$2i+g+c+m$.	$2i+g-c-m$.	$2i+g+c-m$.	$2i+g-c+m$.
4		+·0011	+ ·0002	−·0002
3	−·0010	+·0338	+ ·0052	−·0057
2	−·0285	+·4429	+ ·1331	−·0866
1	−·4227	+·0554	+1·3679	−·0704
0	−·2906	−·0390	+ ·0476	−·0055
−1	+·0182	+·0263	− ·0529	−·0633
−2	−·0111	+·0275	+ ·0025	−·0729
−3	−·0187	+·0011	+ ·0054	−·0051
−4	−·0010		+ ·0001	
Sum ...	−·7554	+·5491	+1·5091	−·3097

$\sqrt{-1}z_{ke^3e'} \div ake^3e'$

i.	$2i+g+3c+m$.	$2i+g-3c-m$.	$2i+g+3c-m$.	$2i+g-3c+m$.
4		+·0001		
3		+·0032		−·0004
2		+·0096	+·0001	+·0023
1	−·0012	+·0274	+·0044	+·0054
0	−·0373	+·0200	+·0485	−·0192
−1	−·0525	+·0029	−·0205	−·0057
−2	+·0002	+·0001	−·0019	−·0002
−3	−·0010			
−4	−·0001			
Sum ...	−·0919	+·0633	+·0306	−·0178

$\sqrt{-1}\, z_{ke^3e'} \div ake^3e'$ (*continued*)

i.	2*i*+g+c+m.	2*i*+g−c−m.	2*i*+g+c−m.	2*i*+g−c+m.
4				
3		+·0010	+·0001	−·0002
2	−·0008	+·0310	+·0038	−·0056
1	−·0253	+·0220	+·0886	−·0219
0	−·0760	+·0329	+·0157	+·0052
−1	−·0152	+·0086	+·0450	−·0258
−2	−·0059	+·0018	+·0012	−·0052
−3	−·0015		+·0004	−·0002
−4				
Sum ...	−·1247	+·0973	+·1548	−·0537

170. *Characteristic* $ke^2e'^2$. Values of

A

i.	2*i*+g+2c+2m.	2*i*+g−2c−2m.	2*i*+g+2c−2m.	2*i*+g−2c+2m.
4		+·0026		
3		+·0576	+ ·0007	−·0039
2	+·0007	+·5520	+ ·0296	−·1331
1	+·0020	−·0432	+ ·4848	−·0045
0	−·2683	+·0063	+ ·7659	+·0171
−1	−·1694	+·0093	− ·0082	−·2014
−2	+·0240	−·0005	− ·0057	−·0288
−3	−·0304		+ ·0036	−·0014
−4	−·0024			
Sum ...	−·4438	+·5841	+1·2707	−·3560

	A				*B*
i.	2*i*+g+2c.	2*i*+g−2c.	2*i*+g+2m.	2*i*+g−2m.	2*i*+g.
4		−·0008		+ ·0003	−·0002
3	−·0003	−·0157		+ ·0172	−·0057
2	−·0109	−·1408	−·0191	+ ·3588	−·1183
1	−·1984	−·0073	−·7306	+2·1857	−·7709
0	−·2647	−·0081	−·0518	− ·0337	+·0233
−1	−·0102	+·1564	+·0428	− ·0030	−·0118
−2	+·0010	+·0163	−·1725	+ ·0199	+·1020
−3	+·0139	+·0006	−·0174	+ ·0005	+·0081
−4	+·0009		−·0006		
Sum...	−·4687	+·0006	−·9492	+2·5457	−·7735

$\sqrt{-1}z_{ke^2e'^2} \div ake^2e'^2$

i.	$2i+g+2c+2m$.	$2i+g-2c-2m$.	$2i+g+2c-2m$.	$2i+g-2c+2m$.
4		+·0001		
3		+·0027		−·0002
2		+·0844	+·0006	−·0159
1	+·0001	+·0735	+·0198	−·13
0	−·0264	+·024	+·0934	−·040
−1	−·2254	+·0011	+·17	−·0280
−2	−·0252		+·011	−·0013
−3	−·0054		+·0005	
−4	−·0001			
Sum ...	−·2824	+·186	+·30	−·22

i.	$2i+g+2c$.	$2i+g-2c$.	$2i+g+2m$.	$2i+g-2m$.	$2i+g$.
4					
3		−·0007		+·0004	−·0001
2	−·0002	−·0188	−·0008	+·0157	−·0048
1	−·0076	+·020	−·0782	+·2968	−·0925
0	−·0287	+·13	−·142	+·089	0
−1	−·032	+·0193	−·067	+·10	+·036
−2	−·001	+·0007	−·0271	+·0027	+·0140
−3	+·0021		−·0008		+·0004
−4					
Sum...	−·067	+·15	−·316	+·50	−·047

$g_{e^2e'^2} = +·0104$

171. *Characteristic* kee'^3. Values of

A

i.	$2i+g+c+3m$.	$2i+g-c-3m$.	$2i+g+c-3m$.	$2i+g-c+3m$.
4		+ ·0004		
3		+ ·0242	+ ·0024	+·0002
2		+ ·4352	+ ·0703	+·0057
1	− ·0012	+1·4451	+ ·8945	−·0051
0	− ·1516	− ·1946	+ ·3422	+·3416
−1	+1·3342	− ·0141	− ·0203	−·2245
−2	− ·1292	− ·0006	− ·0141	−·0671
−3	− ·0256		− ·0005	−·0039
−4	− ·0014			
Sum ...	+1·0252	+1·6956	+1·2745	+·0469

i.	2i+g+c+m.	2i+g−c−m.	2i+g+c−m.	2i+g−c+m.
4		−·0001		
3		−·0102	−·0013	
2	+·0020	−·1742	−·0316	−·0191
1	−·0075	−·6600	−·3931	+·4424
0	−·6609	−·7527	+·7341	+·8012
−1	−·4825	+·0060	+·3980	+·2141
−2	+·1155	−·0035	+·0141	+·0480
−3	+·0152	−·0002	+·0007	+·0022
−4	+·0006		−·0001	
Sum...	−1·0176	−1·5949	+·7208	+1·4888

$\sqrt{-1}z_{kee'^3} \div akee'^3$

i.	2i+g+c+3m.	2i+g−c−3m.	2i+g+c−3m.	2i+g−c+3m.
3		+·0008		
2		+·0341	+·0021	+·0003
1	−·0001	+·7392	+·0634	−·0023
0	−·0367	+·1646	+·1380	−·3056
−1	−1·3081	−·0031	+·0159	−·1227
−2	−·1053		−·0044	−·0054
−3	−·0023			−·0001
Sum...	−1·4525	+·9356	+·2150	−·4358

i.	2i+g+c+m.	2i+g−c−m.	2i+g+c−m.	2i+g−c+m.
3		−·0003		
2	+·0001	−·0124	−·0009	−·0011
1	−·0006	−·2550	−·0253	+·1352
0	−·1712	+·6440	+·2329	−·6877
−1	+·4315	+·0046	−·3420	+·0838
−2	+·0628	−·0002	+·0039	+·0035
−3	+·0012		+·0001	+·0001
Sum...	+·3238	+·3807	−·1313	−·4662

172. *Characteristic* k^3ea. Values of

A

$2i$.	$2i+3g+c$.	$2i+3g-c$.	$2i+g+c$.	$2i+g-c$.
7				+ ·0008
5	+ ·0002	− ·0001	+ ·0026	+ ·0364
3	+ ·0015	+ ·0149	+ ·1586	+1·1788
1	+ ·0171	+ ·9812	+4·6117	− ·1487
−1	+1·5291	+ ·0369	− ·2077	− ·0321
−3	+ ·0723	+ ·0151	+ ·0578	+ ·1668
−5	− ·0063	+ ·0080	+ ·0422	+ ·0068
−7	+ ·0021	+ ·0003	+ ·0017	+ ·0002
Sum...	+1·6480	+1·0563	+4·6669	+1·2090

$\sqrt{-1}\, z_{k^3ea} \div ak^3ea$

$2i$.	$2i+3g+c$.	$2i+3g-c$.	$2i+g+c$.	$2i+g-c$.
7				
5			+ ·0001	+ ·0016
3		+·0006	+ ·0065	+ ·1507
1	+·0019	+·1098	+ ·5230	+ ·966
−1	+·1548	+·171	−1·264	+ ·095
−3	+·128	−·033	− ·090	+ ·0217
−5	+·0065	+·0011	+ ·0059	+ ·0003
−7	+·0004		+ ·0001	
Sum...	+·292	+·250	− ·818	+1·235

173. *Characteristic* $k^3e'a$. Values of

A

$2i$.	$2i+3g+m$.	$2i+3g-m$.	$2i+g+m$.	$2i+g-m$.
7			− ·0003	+ ·0001
5	+·0003	− ·0003	− ·0088	+ ·0078
3	+·0020	+ ·0032	− ·4256	+ ·3012
1	+·0645	+ ·0619	−13·0974	+ 1·5617
−1	+·5923	−1·7294	+ ·9515	−11·1432
−3	−·7884	+ ·3505	+ ·5034	− ·1149
−5	+·0212	− ·0150	+ ·0257	− ·0047
−7	+·0014	− ·0007	+ ·0006	− ·0001
Sum...	−·1067	−1·3298	−12·0509	− 9·3921

$\sqrt{-1}z_{k^3e'a} \div ak^3e'a$

$2i$.	$2i+3g+m$.	$2i+3g-m$.	$2i+g+m$.	$2i+g-m$.
7				
5			− ·0003	+ ·0002
3	+·0001	+·0001	− ·0292	+ ·0209
1	+·0038	+·0034	−3·7306	+ ·5923
−1	+·1406	−·4881	− ·7886	+ 9·4528
−3	+·7385	−·3000	+ ·2256	+ ·0016
−5	+·0192	−·0088	+ ·0021	− ·0002
−7	+·0001	−·0001		
Sum ...	+·9023	−·7935	−4·3210	+10·0676

174. *Characteristic* ke^3a. Values of

A

$2i$.	$2i+g+3c$.	$2i+g-3c$.	$2i+g+c$.	$2i+g-c$.
9		−·0001		
7		−·0089	−·0003	−·0031
5	−·0001	−·0611	−·0122	−·1020
3	−·0141	+·0071	−·3990	−·4204
1	−·5143	−·0045	−·4881	+·0113
−1	+·6224	+·0981	+·0182	−·0063
−3	+·0150	+·0219	−·0105	+·0946
−5	−·0028	+·0006	+·0304	+·0159
−7	+·0029		+·0039	+·0002
−9	+·0004		+·0001	
Sum ...	+·1094	+·0531	−·8575	−·4098

$\sqrt{-1}z_{ke^3a} \div ake^3a$

$2i$.	$2i+g+3c$.	$2i+g-3c$.	$2i+g+c$.	$2i+g-c$.
7		−·0004		−·0001
5		−·0087	−·0003	−·0043
3	−·0003	−·015	−·0157	−·0533
1	−·0191	−·047	−·0554	−·073
−1	+·0641	+·0113	+·111	+·034
−3	+·031	+·0009	+·020	+·0123
−5	+·004		+·0044	+·0007
−7	+·0005		+·0002	
Sum ...	+·080	−·059	+·064	−·084

175. *Characteristic* $ke^2e'a$. Values of

A

$2i$.	$2i+g+2c+m$.	$2i+g-2c-m$.	$2i+g+2c-m$.	$2i+g-2c+m$.	$2i+g+m$.	$2i+g-m$.
9		− ·0002		+ ·0001		
7		− ·0074		+ ·0076	+ ·0018	− ·0010
5	+ ·0047	− ·2310	− ·0004	+ ·1880	+ ·0784	− ·0432
3	+ ·1891	− ·5328	− ·0392	− ·4235	+2·0144	−1·0514
1	+4·7223	+ ·2560	− ·6154	−3·1861	+4·5725	+ ·1211
−1	− ·1328	− ·3295	−2·7916	+ ·2076	− ·0405	+5·5200
−3	− ·4499	− ·1708	− ·2700	+ ·1003	+ ·2134	− ·2235
−5	+ ·0531	− ·0058	− ·0183	+ ·0035	+ ·0769	− ·0745
−7	+ ·0137	− ·0001	− ·0080		+ ·0021	− ·0024
−9	+ ·0003		− ·0002			
Sum...	+4·4005	−1·0216	−3·7431	−3·1025	+6·9190	+4·2451

$\sqrt{-1}z_{ke^2e'a} \div ake^2e'a$

$2i$.	$2i+g+2c+m$.	$2i+g-2c-m$.	$2i+g+2c-m$.	$2i+g-2c+m$.	$2i+g+m$.	$2i+g-m$.
7		−·0002		+ ·0002		
5	+·0001	−·0170		+ ·0124	+ ·0022	− ·0013
3	+·0050	−·2344	−·0011	− ·1335	+ ·1255	− ·0708
1	+·2715	−·2173	−·0391	+2·7063	+1·3020	+ ·0216
−1	−·0299	−·0979	−·8122	+ ·0885	+ ·0199	−4·6849
−3	+·4155	−·0108	+·2425	+ ·0070	+ ·0978	− ·1008
−5	+·0347	−·0002	−·0068	+ ·0001	+ ·0058	− ·0052
−7	+·0011		−·0006		+ ·0001	− ·0001
Sum...	+·6980	−·5778	−·6173	+2·6810	+1·5533	−4·8415

CHAPTER VIII.

TERMS OF THE SIXTH ORDER.

Section (i). *Formulæ and Methods of Procedure for u.*

176. Terms in only two characteristics, e^4k^2, e^2k^4, have been calculated ; for those in e^6, in which very small divisors do not occur, the elliptic values can be substituted ; those in k^6 are insensible. No terms with arguments $2i$ are calculated, as no small divisors are present ; the constant C′ enters only with these terms, and it may therefore be neglected.

The method is that of Chapter VII., with $\Omega_1 = L' = \Lambda' = 0$. Also a takes the values 6, 5, 4, 3, and $b = 6-a$. For $u_3\, u_3$, $s_3\, s_3$, $z_3\, z_3$, $Dz_3\, Dz_3$, $u_3\, s_3 + s_3\, u_3 = 2u_3\, s_3$, &c., we must substitute the halves of these functions when dealing with the general formulæ.

177. The main difference arises in the development of δf, $\delta(Df')$. Here

$$\delta f = c_2\frac{\partial f_4}{\partial c} + g_2\frac{\partial f_4}{\partial g} + \tfrac{1}{2}c_2{}^2\frac{\partial^2 f_2}{\partial c^2} + \tfrac{1}{2}g_2{}^2\frac{\partial^2 f_2}{\partial g^2} + c_4\frac{\partial f_2}{\partial c} + g_4\frac{\partial f_2}{\partial g}\,,$$

with a similar expression for $\delta(Df')$; $\partial^2 f_2/\partial c\partial g$, $\partial^2(Df'_2)/\partial c\partial g$ are zero.

For the first two and last two terms the formulæ of § 134 are available with $a = 4, 3, 2$ and $b = 4-a$, $a=2, 1$ and $b=2-a$; when $a = b$ the remark at the close of § 176 must be noted.

For the other two terms I find, for the coefficients of $\zeta^{\pm(2i+2c)}$ only,

$$\frac{\partial^2 f_2}{\partial c^2} = 6u_e s_e + 8(u_{e^2}s_0 + s_{e^2}u_0)\,,\quad \frac{\partial^2}{\partial c^2}(Df'_2/ = -8(u_{e^2}s_0 - s_{e^2}u_0)\,;$$

$$\frac{\partial^2 f_2}{\partial g^2} = (8u_{k^2}s_0 + s_{k^2}u_0)\qquad,\quad \frac{\partial^2}{\partial g^2}(Df'_2) = -(8u_{k^2}s_0 - s_{k^2}u_0)\,.$$

No general formulæ for the derivatives of Q were obtained, owing to the difficulty of expressing them in convenient forms. The cases are

$a, b = 3, 1$, characteristics ke^2, k ; k^3, k ;

$a, b = 2, 2$, „ ke, ke ;

$a, b = 1, 1$, „ k, k.

For these cases we have, using the notation of § 134 for Σ' and of § 135 for z',

$$a = \mathrm{ke}^2 \text{ or } \mathrm{k}^3, \begin{cases} \dfrac{\partial Q}{\partial c} = 4\Sigma' \dfrac{\partial p}{\partial c} D(z_a z_k) - 2\Sigma' \dfrac{\partial p}{\partial c} z_a D z_k, \\ \dfrac{\partial Q}{\partial g} = \Sigma' D \left\{ 4\dfrac{\partial p}{\partial g} z_a z_k - 2 z_a z'_k \right\} + 4 z_a D z'_k - 2\Sigma' \dfrac{\partial p}{\partial g} z_a D z_k ; \end{cases}$$

$$a = b = ke, \begin{cases} \dfrac{\partial Q}{\partial c} = 2\Sigma' \dfrac{\partial p}{\partial c} D(z^2_{ke}) - 2 D z_{ke} \cdot (z'_{ke})_{g+c} + 2 D z_{ke} \cdot (z'_{ke})_{g-c}, \\ \dfrac{\partial Q}{\partial g} = 2\Sigma' \dfrac{\partial p}{\partial g} D(z^2_{ke}) - 2 z'_{ke} D z_{ke} ; \end{cases}$$

$$a = b = k, \begin{cases} \dfrac{\partial Q}{\partial c} = 0, \quad \dfrac{\partial Q}{\partial g} = 2\Sigma' \dfrac{\partial p}{\partial g} D(z^2_k) - 2 z'_k D z_k, \\ \dfrac{\partial^2 Q}{\partial c^2} = 0, \quad \dfrac{\partial^2 Q}{\partial g^2} = 2\Sigma' \left(\dfrac{\partial p}{\partial g}\right)^2 z^2_k - 2 z'^2_k, \quad \dfrac{\partial^2 Q}{\partial c \partial g} = 0, \end{cases}$$

where $(z'_{ke})_{g+c}$ denotes that the terms whose arguments are $\pm(2i+g+c)$ only are to be used, and similarly for $(z'_{ke})_{g-c}$.

178. The calculations were made on the plan outlined in Section (iv.) of Chapter VII., with certain abbreviations. As in most cases only three significant figures were necessary, it was found possible, in forming the products of series, for the computer to add each pair of logarithms from the slips and look out the number corresponding to their sum from a four-place table (printed on a card) without writing anything down (see § 140). Only this last number was actually written down, and thus the sheets on which the various series were added together could be entered straight from the multiplication slips. The great majority of the products consisted of numbers with one or two significant figures, and long practice has made us so familiar with the logarithms of numbers from 1 to 99 that a glance at the table was rarely necessary for these.

Nearly all the other operations were so arranged that the use of logarithms was not necessary and the computation sheets were much abbreviated. In the few cases where logarithms had to be used they were written down on a spare corner of the sheet, so as not to disturb the general plan.

Only those sets of coefficients corresponding to the arguments $2i\pm(2c-2g)$ required the special method of § 138.

Section (ii). *The Homogeneous Equation for z.*

179. As the calculation of B_6 (see § 139) would have been long, the homogeneous equation (8) of § 7 was used. The terms calculated were those with characteristics k^5e, k^3e^3, ke^5, so that we have $\Omega_1=0$. The equation is then

$$D(uDz-zDu)-2mzDu-m^2uz-\frac{3}{2}m^2z(u+s)=0.$$

Put $u'=u\zeta^{-1}$, $s'=s\zeta$, and divide by ζ. The equation may be written

$$f''+\delta f''\equiv\left(D^2-1-2m-\frac{5}{2}m^2\right)zu'-2(D+1+m)zDu'-\frac{3}{2}m^2zs'\zeta^{-2}=0, \qquad \ldots \qquad \ldots \qquad (1)$$

or

$$(D^2-g_0^2)zu'-2(D+g_0)zDu'+(g_0^2-1-2m-\tfrac{5}{2}m^2)zu'+2(g_0-1-m)zDu'-\tfrac{3}{2}m^2zs'\zeta^{-2}=0. \quad (2)$$

The constant coefficients in the last three terms of this equation are small. In fact,

$$g_0^2-1-2m-\tfrac{5}{2}m^2=-\cdot 00044, \quad g_0-1-m=+\cdot 00432, \quad \tfrac{3}{2}m^2=+\cdot 009805. \quad \ldots \quad (3)$$

Also, since Du_0' has the factor m^2, the term of principal importance in the determination of z_6 is the first. If then the operation $(D^2-g_0^2)^{-1}$ be performed on the equation, the first approximation to z_6 is given by

$$az_6=-zu'+\left(\frac{2}{D-g_0}-\frac{2(g_0-1-m)}{D^2-g_0^2}\right)zDu'+\frac{1+2m+\frac{5}{2}m^2-g_0^2}{D^2-g_0^2}zu'+\frac{1}{D^2-g_0^2}\frac{3}{2}m^2zs'\zeta^{-2}=0, \quad \ldots \quad (4)$$

where we neglect z_6 on the right. For the second approximation we substitute $-z_6^{(1)}(u_0'-a)$ (§ 132) for the first term on the right and $z_6^{(1)}$, Du_0' for z, Du' in the other terms; and so on.

180. As in Chapter VII., c_0, g_0 are to be used for c, g in performing operations involving D. Let $\delta f''$ denote the parts of (1) due to $c-c_0$, $g-g_0$. Then

$$\delta f''=c_2\frac{\partial f_4''}{\partial c}+g_2\frac{\partial f_4''}{\partial g}+c_4\frac{\partial f_2''}{\partial c}+g_4\frac{\partial f_2''}{\partial g};$$

the terms involving second order derivatives are found to be negligible, owing to the smallness of c_2^2, c_2g_2, g_2^2.

Let q denote an index of ζ in u'. Then, using the notations of §§ 134, 135,

$$\frac{\partial f''}{\partial c}=\Sigma'\frac{\partial p}{\partial c}\left\{2D(zu')-2zDu'\right\}-2(D+1+m)z\Sigma'\frac{\partial q}{\partial c}u'.$$

Since $1+m-g_0$ is small enough to be neglected when multiplied by $c-c_0$, $g-g_0$ (there are no small divisors), this may be written:

$$\frac{\partial f''}{\partial c}=2\Sigma'\frac{\partial p}{\partial c}u'Dz-2(D+g_0)z\Sigma'\frac{\partial q}{\partial c}u',$$

with a similar expression for $\partial f''/\partial g$.

Hence, after putting $c=c_0$, $g=g_0$ in the coefficients of ζ in (4), we must add

$$-\frac{1}{D^2-g_0^2}\delta f''$$

to its right-hand member; this expression separates into two parts, involving the operators $(D^2-g_0^2)^{-1}$ and $(D-g_0)^{-1}$ respectively.

181. The actual computations are comparatively short. The products

$$z_au'_b,\ z_aDu'_b, \text{ with } a, b = 5, 1;\ 4, 2;\ 3, 3;\ 3, 1;\ 2, 2;\ 1, 1,$$

are obtained according to the plan explained in § 140, and the other remarks there made apply here also. Moreover, as there are no monthly terms, small divisors are not

present. In most cases the terms multiplied by (3) can be altogether neglected; when they are not quite insensible a simple inspection shows what additions arise from them. It was also found that a second approximation to the value of z_6 was not necessary.

Section (iii). *Values of* $A_\lambda/a^2\lambda$, $B_\lambda/a^2\lambda$, $u_\lambda\zeta^{-1}/a\lambda$, $\iota z/a\lambda$.

182. The tables giving the characteristics and arguments calculated are (see § 148):—

§.	λ.	Arguments.
183	e^4k^2	$2i\pm4c\pm2g$, $2i\pm4c$, $2i\pm2c\pm2g$, $2i\pm2c$, $2i\pm2g$
184	e^2k^4	$2i\pm2c\pm4g$, $2i\pm2c\pm2g$, $2i\pm4g$, $2i\pm2c$, $2i\pm2g$

§.	λ.	Arguments.
185	k^5e	$2i\pm5g\pm c$, $2i\pm3g\pm c$, $2i\pm g\pm c$
186	k^3e^3	$2i\pm3g\pm3c$, $2i\pm3g\pm c$, $2i\pm g\pm3c$, $2i\pm g\pm c$
187	ke^5	$2i\pm g\pm5c$, $2i\pm g\pm3c$, $2i\pm g\pm c$

The terms for which the method of § 138 was necessary, owing to small divisors, were

Arguments.	Characteristics.
$\pm(2c-2g)$,	e^4k^2, e^2k^4.

In §§ 183, 184 the arrangement of §§ 149–163 is followed exactly. In §§ 185–187 the final results for $\iota z_\lambda/a\lambda$ are alone given, as there was no definite stopping place in the computations, and the first approximations are the final results.

183. *Characteristic* e^4k^2. Values of

A

i	$2i+4c+2g$.	$2i+4c-2g$.	$2i+4c$.	$2i+2c+2g$.	$2i+2c-2g$.	$2i+2c$.	$2i+2g$.
3					−·0009		
2		−·002	−·001	+·002	−·0233		−·014
1	+·002	−·045	+·007	−·021	−·0157	+·101	−·371
0	+·005	−·036	+·354	−·707	−·1346	+·052	+·059
−1	−·079	+·008	−·052	+·133	+·0089	−·019	−·015
−2	+·010	−·001	−·004	+·001	−·0674	−·003	−·004
−3		−·006		−·002	−·0025	−·008	−·005
Sum...	−·062	−·082	+·304	−·594	−·2355	+·123	−·350

B

i.	$2i+4c+2g$.	$2i+4c-2g$	$2i+4c$.	$2i+2c+2g$.	$2i+2c-2g$.	$2i+2c$	$2i+2g$.
3					+·0002		
2					+·0072	+·001	−·004
1		+·012	−·001	−·020	−·0493	−·066	+·084
0	−·031	−·073	−·179	+·142	−·0054	+·104	−·134
−1	+·027	·000	+·043	−·058	−·0037	−·010	+·003
−2	−·007	+·003	+·002	+·003	+·0196	+·004	−·008
−3			+·001	−·002	−·0005	+·004	+·002
Sum...	−·011	−·058	−·134	+·065	−·0319	+·037	−·057

$u_{e^4k^2}\zeta^{-1} \div ae^4k^2$

i.	$2i+4c+2g$.	$2i-4c-2g$.	$2i+4c-2g$.	$2i-4c+2g$.	$2i+4c$.	$2i-4c$.
3				−·001		−·001
2		+·002		+·003		−·01
1		−·006	−·003	−·01	+·001	−·001
0	+·010	+·011	−·044	+·031	+·059	+·027
−1	−·028	+·001	+·02	−·007	−·026	−·001
−2	+·010		−·004		+·01	
−3	·00				−·002	
Sum ...	−·01	+·008	−·03	+·02	+·04	+·01

$u_{e^4k^2}\zeta^{-1} \div ae^4k^2$ *(continued)*

i.	$2i+2c+2g$.	$2i-2c-2g$.	$2i+2c-2g$.	$2i-2c+2g$.	$2i+2c$.	$2i-2c$.	$2i+2g$.	$2i-2g$.
3		−·001				−·003		+·001
2	+·001	−·01	−·002	−·015	−·001	−·004		−·006
1	+·006	+·024	−·025	+·032	+·022	+·01	−·078	−·06
0	−·145	−·022	+·64	−·58	−·056	+·102	+·086	+·064
−1	+·083	+·007	+·003	+·017	·00	+·001	+·03	−·016
−2	·00	+·001	−·004	−·004	−·002		+·004	+·003
−3			+·001		−·004		−·002	
Sum...	−·06	·00	+·61	−·55	−·04	+·11	+·04	−·01

184. *Characteristic* e^2k^4. Values of

A

i.	$2i+2c+4g$.	$2i+2c-4g$.	$2i+2c+2g$.	$2i+2c-2g$.	$2i+4g$.	$2i+2c$.	$2i+2g$.
3				+ ·0001			
2		+·003		− ·0051	+·001	− ·001	− ·003
1		+·178	+·002	− ·3548	−·004	+ ·010	− ·058
0		+·810	+·039	−5·1477	−·063	+1·696	−5·159
−1	−·002	+·003	−·138	− ·5573	+·299	+ ·506	−1·219
−2	+·017	−·002	+·010	− ·0056	+·027	− ·010	− ·039
−3			−·002	− ·0001	−·001	− ·001	
Sum...	+·015	+·992	−·089	−6·0705	+·259	+2·200	−6·478

B

i.	$2i+2c+4g$.	$2i+2c-4g$.	$2i+2c+2g$.	$2i+2c-2g$.	$2i+4g$.	$2i+2c$.	$2i+2g$.
3							
2		+·006		−·0028		− ·001	−·003
1		+·002	−·008	−·1373	+·004	− ·014	−·215
0	−·001	+·369	−·665	+·0896	+·267	−1·661	−·802
−1	+·056	+·058	+·268	−·0743	+·013	− ·148	+·062
−2	−·010	+·001	−·001	−·0148	−·001	+ ·027	−·004
−3				−·0003		− ·002	−·002
Sum...	+·045	+·436	−·406	−·1399	+·283	−1·799	−·964

$u_{e^2k^4}\zeta^{-1} \div ae^2k^4$

·	$2i+2c+4g$.	$2i-2c-4g$.	$2i+2c-4g$.	$2i-2c+4g$.	$2i+2c+2g$.	$2i-2c-2g$.
3		·00				−·001
2		+·005	+·001			·00
1		−·042	−·02	+·002	+·001	−·029
0		+·001	−·030	+·334	+·003	+·272
−1			−·073	−·08	−·036	+·002
−2	+·010		−·001	−·001	−·01	
−3	·00				+·002	
Sum...	+·01	−·04	−·12	+·25	−·04	+·24

i.	$2i+2c-2g$.	$2i-2c+2g$.	$2i+4g$.	$2i-4g$.	$2i+2c$.	$2i-2c$.	$2i+2g$.	$2i-2g$.
3								−·001
2	−·002	−·002		·00		−·001		−·029
1	−·182	−·210	−·001	+·065	+·001	+·08	−·015	−·16
0	−·98	+2·55	−·002	−·353	+·443	+1·725	−1·818	−·226
−1	−·117	−·058	+·144	−·005	−·19	−·009	+·78	−·002
−2	−·013	−·003	−·02		−·006		−·021	−·001
−3					+·001		+·001	
Sum...	−1·29	+2·28	+·12	−·29	+·25	+1·79	−1·07	−·42

185. *Characteristic* k^5e. Value of $\sqrt{-1}\,z_{k^5e} \div ak^5e$.

i.	$2i+5g+c$.	$2i+5g-c$.	$2i+3g+c$.	$2i+3g-c$.	$2i+g+c$.	$2i+g-c$.
2						
1				$-$ ·01		$-$ ·11
0		$-$·01		$-$2·45	+·82	$-$9·81
$-$1		+·08	$-$·07	+ ·61	+·14	+ ·24
$-$2	+·01	+·01	$-$·01			$-$ ·01
Sum...	+·01	+·08	$-$·08	$-$1·85	+·96	$-$9·69

186. *Characteristic* k^3e^3. Value of $\sqrt{-1}\,z_{k^3e^3} \div ak^3e^3$.

i.	$2i+3g+3c$.	$2i+3g-3c$.	$2i+3g+c$.	$2i+3g-c$.	$2i+g+3c$.	$2i+g-3c$.	$2i+g+c$.	$2i+g-c$.
3						$-$·01		
2		$-$·01				$-$·03		
1		+·01		$-$·10		+·09		$-$·02
0		+·06	$-$·25	+·09	+·12	$-$·16	$-$·03	$-$·19
$-$1	$-$·03	+·01	+·10	+·03	$-$·03	+·02	+·01	$-$·04
$-$2			+·02				+·01	
$-$3								
Sum...	$-$·03	+·07	$-$·13	+·02	+·09	$-$·09	$-$·01	$-$·25

187. *Characteristic* ke^5. Value of $\sqrt{-1}\,z_{ke^5} \div ake^5$.

i.	$2i+g+5c$.	$2i+g-5c$.	$2i+g+3c$.	$2i+g-3c$.	$2i+g+c$.	$2i+g-c$.
3				+·001		+·001
2			+·001	$-$·001	+·004	·000
1		+·002	+·010	+·003	$-$·002	·000
0	+·010	$-$·004	$-$·007	$-$·002	$-$·001	$-$·005
$-$1	$-$·009	$-$·001	+·001	$-$·004	+·002	$-$·001
$-$2	+·002		+·004		$-$·001	$-$·001
$-$3						
Sum...	+·003	$-$·003	+·009	$-$·003	+·002	$-$·006

CHAPTER IX.

Results in Polar Coordinates.

Section (i). *Formulæ for Transformation.*

188. *Longitude.*—The formulæ are given in § 41. The value of V_0 is obtained by special values from equation (48) and the other V_μ from equation (47) in that section. By development, using the notation of § 139, we obtain

$$2\iota V_1 = (u_1 - s_1) \div (u_0,\ s_0),$$
$$2\iota V_2 = (u_2 - s_2 - \tfrac{1}{2}u_1^2 + \tfrac{1}{2}s_1^2) \div (u_0,\ s_0),$$
$$2\iota V_3 = (u_3 - s_3 - u_1u_2 + s_1s_2 + \tfrac{1}{3}u_1^3 - \tfrac{1}{3}s_1^3) \div (u_0,\ s_0),$$
$$2\iota V_4 = (\mathrm{M} - \overline{\mathrm{M}}) \div (u_0,\ s_0),$$
$$2\iota V_5 = (u_5 - s_5 - u_2u_3 + s_2s_3 - u_1\mathrm{M}' + s_1\overline{\mathrm{M}}') \div (u_0,\ s_0),$$
$$2\iota V_6 = \{u_6 - s_6 - (u_1u_5 + u_2u_4 + \tfrac{1}{2}u_3^2) + (s_1s_5 + s_2s_4 + \tfrac{1}{3}s_3^2) + u_1^2\mathrm{M}' - s_1^2\overline{\mathrm{M}}' + u_2\mathrm{N} - s_2\overline{\mathrm{N}}\} \div (u_0,\ s_0),$$

where

$$\mathrm{M} \div (u_0,) = \frac{u_4}{u_0} - \frac{1}{2}\frac{2u_1u_3 + u_2^2}{u_0^2} + \frac{u_1^2u_2}{u_0^3} - \frac{1}{4}\frac{u_1^4}{u_0^4},$$

$$\mathrm{M}' \div (u_0,) = \mathrm{M} \div (u_0,) - \frac{1}{2}\frac{u_2^2}{u_0^2} + \frac{1}{20}\frac{u_1^4}{u_0^4}, \qquad \mathrm{N} \div (u_0,) = \frac{u_2^2 + 6u_1u_3}{3u_0^2} - \frac{u_1^2u_2}{2u_0^3}.$$

All the products and most of the sums in V_1, V_2, V_3, V_4, M, M′, N, were at hand. For V_5 the product $u_2\,u_3$ was available; the only products to be formed were, therefore, $u_2\,u_3$ by $1/u_0^2$ and $\mathrm{M}' \div (u_0,)$ by u_1/u_0. In V_6 the known factor $u_1\,u_5 + u_2\,u_4 + \frac{1}{2}u_3^2$ was multiplied by $1/u_0^2$, u_1^2/u_0^2 by $\mathrm{M}' \div (u_0,)$ and u_2/u_0 by $\mathrm{N} \div (u_0,)$. In all cases the corresponding functions of s were obtained by putting $1/\zeta$ for ζ in u.

189. *Parallax.*—For convenience in obtaining the latitude, equation (49), § 42, for the parallax was computed in the form

$$\frac{\rho_0}{r} = 1 + \left(\frac{\rho_0}{r}\right)_1 + \left(\frac{\rho_0}{r}\right)_2 + \left(\frac{\rho_0}{r}\right)_3 + \left(\frac{\rho_0}{r}\right)_4$$

for orders up to the fourth inclusive. Here

$$\left(\frac{\rho_0}{r}\right)_1 = -\tfrac{1}{2}(u_1 + s_1) \div (u_0,\ s_0),$$
$$\left(\frac{\rho_0}{r}\right)_2 = \{-\tfrac{1}{2}(u_2 + s_2) + \tfrac{3}{8}(u_1^2 + s_1^2) + \tfrac{1}{4}u_1s_1 - \tfrac{1}{2}z_1^2\} \div (u_0,\ s_0,\ \rho_0),$$
$$\left(\frac{\rho_0}{r}\right)_3 = \{-\tfrac{1}{2}(u_3 + s_3) + \tfrac{3}{4}(u_1u_2 + s_1s_2) + \tfrac{1}{4}(u_1s_2 + s_1u_2) - z_1z_2 - \tfrac{5}{16}(u_1^3 + s_1^3) - \tfrac{3}{16}(u_1^2s_1 + s_1^2u_1) + \tfrac{3}{4}z_1^2(u_1 + s_1)\} \div (u_0,\ s_0,\ \rho_0),$$
$$\begin{aligned}\left(\frac{\rho_0}{r}\right)_4 = \{&-\tfrac{1}{2}(u_4 + s_4) + \tfrac{3}{4}(u_1u_3 + s_1s_3) + \tfrac{3}{8}(u_2^2 + s_2^2) + \tfrac{1}{4}(u_1s_3 + s_1u_3 + u_2s_2) - z_1z_3 - \tfrac{1}{2}z_2^2 - \tfrac{15}{16}(u_1^2u_2 + s_1^2s_2) \\ &- \tfrac{3}{16}(2u_1u_2s_1 + 2s_1s_2u_1 + u_1^2s_2 + s_1^2u_2) + \tfrac{3}{4}z_1^2(u_2 + s_2) + \tfrac{3}{2}z_1z_2(u_1 + s_1) + \tfrac{35}{128}(u_1^4 + s_1^4) \\ &+ \tfrac{5}{32}(u_1^3s_1 + s_1^3u_1) + \tfrac{9}{64}u_1^2s_1^2 - \tfrac{15}{16}z_1^2(u_1^2 + s_1^2) - \tfrac{9}{8}z_1^2u_1s_1 + \tfrac{3}{8}z_1^4\} \div (u_0,\ s_0,\ \rho_0).\end{aligned}$$

All the products and, by suitable rearrangements which differed with different characteristics, many of the sums were at hand.

For the terms of the fifth order, in which the characteristics e^5, $\mathrm{e}^3\mathrm{k}^2$, ek^4 were alone needed, it was shorter to use the Jacobian integral, equation (5), § 7. Here $\Omega_1=0$ and C' is not present in odd order terms, so that

$$2\left(\frac{\kappa}{r}\right)_5=\left[-Du\,.\,Ds-(Dz)^2-\tfrac{3}{4}\mathrm{m}^2(u+s)^2+\mathrm{m}^2z^2\right]_5.$$

All the products had been obtained in finding u_5. An examination of the errors produced by using c_0, g_0 for c, g showed that they were insensible. The terms factored by m^2 are also insensible in most cases.

The order of accuracy for the parallax is found by dividing the characteristics required in the latitude by k.

190. *Latitude.*—To the sixth order inclusive this is given by (§ 43)

$$\phi=\frac{z}{\rho}-\frac{1}{3}\frac{z^3}{\rho^3}+\frac{1}{5}\frac{z^5}{\rho^5}=\frac{z}{r}+\frac{1}{6}\frac{z^3}{\rho^3}-\frac{7}{40}\frac{z^5}{\rho^5}$$
$$=\sum_{j=1}^{j=6}\left(\sum_{i=1}^{i=6}\frac{z_i}{\rho_0}\mathrm{R}_{j-i}\right),$$

where

$$\mathrm{R}_1=\left(\frac{\rho_0}{r}\right)_1,\quad \mathrm{R}_2=\left(\frac{\rho_0}{r}\right)_2+\frac{1}{6}\frac{z_1^2}{\rho_0^2},\quad \mathrm{R}_3=\left(\frac{\rho_0}{r}\right)_3+\{\tfrac{1}{3}z_1z_2-\tfrac{1}{4}z_1^2(u_1+s_1)\}\div(u_0,\,s_0,\,\rho_0),$$

$$\mathrm{R}_4=\left(\frac{\rho_0}{r}\right)_4+[\tfrac{1}{3}z_1z_3+\tfrac{1}{6}z_2^2-\tfrac{1}{2}z_1z_2(u_1+s_1)+z_1^2\{-\tfrac{1}{4}(u_2+s_2)+\tfrac{5}{16}(u_1^2+s_1^2)+\tfrac{3}{8}u_1s_1\}-\tfrac{7}{40}z_1^4]\div(u_0,\,s_0,\,\rho_0),$$

$$\frac{\mathrm{R}_5}{\rho_0}=\left(\frac{1}{r}\right)_5+\frac{1}{6}\left(\frac{z^2}{r^3}\right)_5+\frac{3}{40}\left(\frac{z^4}{\rho^5}\right)_5$$
$$=\left(\frac{1}{r}\right)_5+\frac{1}{12\kappa}\{D^2(z^2)-(Dz)^2\}_5+\frac{1}{\rho_0}\cdot\frac{z_1^2}{\rho_0^2}\left\{\frac{3}{10}\frac{z_1z_2}{\rho_0^2}-\frac{3}{16}\frac{z_1^2}{\rho_0^2}\left(\frac{u_1}{u_0}+\frac{s_1}{s_0}\right)\right\}.$$

The expression for $(z^2/r^3)_5$ in R_5/ρ_0 is obtained by multiplying equation (4), § 6, by z, putting $\Omega_1=0$ and neglecting the terms factored by m^2; we can also put $\mathrm{c}=\mathrm{c}_0$, $\mathrm{g}=\mathrm{g}_0$ in this expression.

All the products in the R_{j-i} were known except those in the last term of R_5/ρ_0. When the R_{j-i} have been obtained the multiplications by z_i/ρ_0 were straightforward.

Section (ii). *Change of the Arbitrary Constants.*

191. The usual constants e, γ, a used in the lunar theory are so defined that the coefficient of the principal elliptic term in longitude is

$$2e-\tfrac{1}{4}e^3+\tfrac{5}{96}e^5;$$

that of the principal term in latitude,

$$2\gamma-2\gamma e^2-\tfrac{1}{4}\gamma^5+\tfrac{7}{32}\gamma e^4;$$

and a is defined by the equation

$$a^3n^2=E+M.$$

The results of the transformation to polar coordinates furnish the following coefficients for these two terms :—

$$+\cdot99972\ 871\ \mathrm{e}+\cdot01457\ 76\ \mathrm{e}^3-\cdot02435\ \mathrm{e}e'^2+1\cdot00499\ 9\ \mathrm{ek}^2-\cdot0171\ \mathrm{e}\alpha^2$$
$$+\cdot0021\ \mathrm{e}^5-\cdot046\ \mathrm{e}^3e'^2+\cdot3714\ \mathrm{e}^3\mathrm{k}^2+\cdot23\ \mathrm{e}e'^2\mathrm{k}^2+1\cdot552\ \mathrm{ek}^4,$$

and

$$+1{\cdot}99974\ 473\ \mathrm{k}+{\cdot}99167\ 5\mathrm{k}^3-{\cdot}25024\ 2\ \mathrm{k}\mathrm{e}^2+{\cdot}00073\ \mathrm{k}e'^2-{\cdot}0098\ \mathrm{k}\alpha^2$$
$$+1{\cdot}48\ \mathrm{k}^5-{\cdot}795\ \mathrm{k}^3\mathrm{e}^2+{\cdot}0\ \mathrm{k}^3e'^2-{\cdot}0287\ \mathrm{k}\mathrm{e}^4-{\cdot}05\ \mathrm{k}\mathrm{e}^2e'^2\,;$$

also (§ 44)

$$\mathrm{a}={\cdot}99909\ 31420\,a.$$

192. Equating these, I find

$$\mathrm{e}=+2{\cdot}00054\ 273\ e-{\cdot}36681\ 52\ e^3+{\cdot}04873\ ee'^2-2{\cdot}01160\ 2\ e\gamma^2+{\cdot}0342\ e\left(\frac{a}{a'}\right)^2$$
$$+{\cdot}049\ e^5+{\cdot}35\ e^3e'^2-{\cdot}246\ e^3\gamma^2-{\cdot}56\ ee'^2\gamma^2+{\cdot}911\ e\gamma^4,$$
$$\mathrm{k}=+1{\cdot}00012\ 765\ \gamma-{\cdot}49609\ 1\ \gamma^3-{\cdot}49924\ 3\gamma e^2-{\cdot}00037\ \gamma e'^2+{\cdot}0049\ \gamma\left(\frac{a}{a'}\right)^2$$
$$-{\cdot}128\ \gamma^5+1{\cdot}07\ \gamma^3e^2+{\cdot}0\ \gamma^3e'^2-{\cdot}095\ \gamma e^4+{\cdot}12\ \gamma e^2e'^2,$$
$$\mathrm{a}=+{\cdot}99909\ 314\ a, \qquad \frac{1}{\mathrm{a}}=+1{\cdot}00090\ 768\,\frac{1}{a}.$$

Section (iii). *Numerical Values of the Constants.*

193. The following are the values of the constants used in reducing the results to seconds of arc :—

$$n=173\ 25594''{\cdot}06, \qquad n'=12\ 95977''{\cdot}415, \qquad \mathrm{m}={\cdot}08084\ 89338,$$
$$e={\cdot}05490\ 056, \qquad e'={\cdot}01677\ 191, \qquad \gamma={\cdot}04488\ 716,$$
$$\frac{1}{a}=3419''{\cdot}596, \qquad \frac{1}{a'}=8''{\cdot}7800, \qquad \frac{E}{M}=81{\cdot}500,$$

giving

$$\alpha_1=\frac{a}{a'}\cdot\frac{E-M}{E+M}={\cdot}00250\ 532.$$

The value of e corresponds to a coefficient $22639''{\cdot}580$ of the principal elliptic term in longitude ; that of γ to a coefficient $18461''{\cdot}480$ of the principal term in latitude ; that of a to the value $3422''{\cdot}700$ for the constant term in the sine of the equatorial horizontal parallax of the Moon.

Section (iv). *Numerical Values of the Parts of the Arguments and Coefficients arising from the Various Characteristics.*

194. The coefficient of each periodic term in longitude and latitude is of the form $\lambda P(e^2, e'^2, \gamma^2, \alpha_1^2)$, where λ is the characteristic of the principal part and the factor of λ is a quadruple power series proceeding according to powers of $e^2, e'^2, \gamma^2, \alpha_1^2$, with numerical coefficients ; in the sine of the parallax each coefficient is of the same form with the additional factor $1/a$. In the longitude and parallax only even powers, and in the latitude only odd powers, of γ are present. In this section will be given the coefficients of all those periodic terms for which P is not limited by the calculations to a single term. The part due to each characteristic in a coefficient of a given periodic term is separately shown. Those periodic terms which have had the parts due to the principal characteristics alone calculated are not set down in this section, since these parts are the final values for these coefficients given in the next section.

The characteristics in all cases are composed of the new constants e, e', γ, α_1, that

is to say, they are the same as those of Delaunay, with the exception a_1 for a. A direct comparison of each part with the corresponding part given by Delaunay is thus possible, when allowance has been made for the slight difference in the numerical values for the constants used by Delaunay and myself, and for the change from a to a_1.

195. The following table gives the various parts of the annual mean motions of the perigee and node * due to the separate characteristics set down in the first column.

Annual Mean Motions.

Char.	Perigee.	Node.
1	$+148524''\cdot92$	$-69287''\cdot90$
e^2	$-\quad 519\cdot31$	$-\quad 616\cdot09$
e'^2	$+\quad 156\cdot27$	$-\quad 25\cdot46$
γ^2	$-\quad 1739\cdot85$	$+\quad 260\cdot59$
a_1^2	$+\quad 2\cdot24$	$-\quad 1\cdot11$
e^4	$+\quad \cdot04$	$+\quad \cdot07$
$e^2e'^2$	$-\quad \cdot99$	$-\quad \cdot57$
$e^2\gamma^2$	$+\quad 6\cdot72$	$-\quad 1\cdot70$
$e'^2\gamma^2$	$-\quad 1\cdot61$	$+\quad \cdot08$
γ^4	$-\quad 1\cdot51$	$+\quad \cdot05$
Sum...	$+146426\cdot92$	$-69672\cdot04$

196. In §§ 197–262 are given the coefficients of those periodic terms in longitude, latitude, and parallax which contain more than one characteristic. The arrangement is primarily according to the orders of the principal characteristics, each principal characteristic being attached to a definite set of arguments which differ only by multiples of 2D and in the signs of the multiples of l, l', F.

The notation for the arguments is that of Delaunay (§ 10).

In the first column of each Table is placed the characteristic, and in the succeeding columns the coefficients corresponding to the multiple (i) of 2D placed in the first row of each Table.

In the last row is given the sum for each column, and therefore the final value for the coefficient of each argument.

The coefficients in longitude are given in §§ 197–225, in units of $0''\cdot001$.

The coefficients in latitude are given in §§ 226–248, in units of $0''\cdot001$.

The coefficients in parallax are given in §§ 249–262, in units of $0''\cdot0001$.

All coefficients have been calculated so as to be correct to the last figure given, with the exception of those depending on e^6, for which the elliptic values have been substituted. This remark applies also to the Tables in section (v).

* I have obtained the complete theoretical values of these two quantities and compared them with their observational values in the first of the two papers referred to in § 147.

197. Arg. $2i$D.

i.		3	2	1
1		+ 49	+ 8740	+2106246
e^2	+1	+ 67	+ 5217	+ 298973
e'^2			− 30	− 1993
γ^2		− 3	− 309	− 31435
$\alpha_1{}^2$			+ 2	+ 37
e^4		+ 17	+ 433	− 343
$e^2e'^2$			− 16	− 191
e'^4				0
$e^2\gamma^2$		− 3	− 136	− 1311
$e'^2\gamma^2$				+ 35
γ^4			+ 1	− 121
$e^2\alpha_1{}^2$				+ 4
$\gamma^2\alpha_1{}^2$				− 2
Sum ...	+1	+127	+13902	+2369899

198. Arg. $l+2i$D.

i.	3	2	1	0	−1	−2	−3	−4
e	+12	+1446	+174865	+22648107	−4608089	−35221	−291	−2
e^3	+10	+ 574	+ 20813	− 8533	+ 1231	− 4143	−114	−2
ee'^2		− 6	− 238	0	+ 1586	+ 77	+ 2	
$e\gamma^2$	− 1	− 58	− 3314	0	+ 18897	+ 811	+ 13	
$e\alpha_1{}^2$			+ 4	0	− 50			
e^5	+ 2	+ 42	− 44	+ 5	+ 4	+ 4	− 8	
$e^3e'^2$		− 2	− 21	0	+ 13	+ 8	+ 1	
$e^3\gamma^2$		− 17	− 115	0	− 19	+ 38	+ 4	
$ee'^2\gamma^2$			+ 6	0	+ 15	− 2		
$e\gamma^4$			− 2	0	− 26			
Sum...	+23	+1979	+191954	+22639579	−4586438	−38428	−393	−4

199. Arg. $l'+2iD$.

i.	3	2	1	0	−1	−2	−3
e'	−1	−180	−21595	−659271	−152090	−1255	−10
e^2e'	−2	−113	− 3476	− 15490	− 15125	− 636	−13
e'^3			+ 1	− 5	+ 118	+ 3	
$e'\gamma^2$		+ 10	+ 651	+ 5702	+ 1651	+ 38	+ 1
$e'\alpha_1^2$			− 7				
e^4e'	−1	− 10	− 26	+ 27	+ 3	− 44	− 3
$e^2e'^3$			+ 1	− 22	+ 8	+ 1	
$e^2e'\gamma^2$		+ 4	− 2	+ 120	+ 82	+ 14	+ 1
$e'^3\gamma^2$					− 2		
$e'\gamma^4$			+ 2	− 5	+ 4		
Sum...	−4	−289	−24451	−668944	−165351	−1879	−24

200. Arg. $2i_1D$.

$2i_1$.	5	3	1
α_1	+8	+735	−125394
$e^2\alpha_1$	−3	−383	− 2433
$e'^2\alpha_1$		+ 7	− 21
$\gamma^2\alpha_1$		+ 42	+ 3040
α_1^3			− 1
$e^4\alpha_1$	−1	− 9	+ 5
$e^2\gamma^2\alpha_1$		+ 10	+ 36
$\gamma^4\alpha_1$			− 17
Sum ...	+4	+402	−124785

201. Arg. $2l+2iD$.

i.	3	2	1	0	−1	−2	−3	−4
e^2	+2	+169	+13241	+771167	−212622	−31054	−531	−7
e^4	+2	+ 56	+ 1478	− 1038	+ 92	− 31	− 59	−2
$e^2e'^2$		− 1	− 25	− 59	− 69	+ 36	+ 3	
$e^2\gamma^2$		− 8	− 299	− 1037	+ 950	+ 279	+ 16	
$e^2\alpha_1^2$				− 1	− 2	− 2		
e^6				+ 1				
$e^4\gamma^2$		− 3	− 9	+ 3	− 2		+ 1	
$e^2\gamma^4$			+ 1	− 15	− 5	− 1		
Sum...	+4	+213	+14387	+769021	−211658	−30773	−570	−9

202. Arg. $l+l'+2i\mathrm{D}$.

$i.$	2	1	0	−1	−2	−3	−4
ee'	−37	−2662	−110214	−206896	−4088	−55	−1
e^3e'	−16	− 356	− 660	− 149	− 393	−19	
ee'^3			− 84	+ 87	+ 6		
$ee'\gamma^2$	+ 2	+ 94	+ 1154	+ 740	+ 79	+ 2	
$ee'\alpha_1{}^2$		− 1		− 1			
Sum ...	−51	−2925	−109804	−206219	−4396	−72	−1

203. Arg. $l-l'+2i\mathrm{D}$.

$i.$	3	2	1	0	−1	−2	−3
ee'	+3	+216	+13634	+149260	+27878	+578	+ 8
e^3e'	+2	+ 74	+ 1177	+ 61	+ 302	+ 74	+ 3
ee'^3			− 13	+ 104	− 54	+ 1	
$ee'\gamma^2$		− 7	− 203	− 1549	+ 385	− 16	
$ee'\alpha_1{}^2$				+ 2			
Sum ...	+5	+283	+14595	+147878	+28511	+637	+11

204. Arg. $2l'+2i\mathrm{D}$.

$i.$	2	1	0	−1	−2	−3
e'^2	+1	− 12	−7313	−7602	−107	−1
$e^2e'^2$	−4	−182	− 260	− 586	− 48	−1
e'^4			+ 2	+ 6		
$e'^2\gamma^2$		+ 5	+ 66	+ 66	+ 4	
Sum ...	−3	−189	−7505	−8116	−151	−2

205. Arg. $2F + 2iD$.

$i.$	3	2	1	0	−1	−2	−3
γ^2		−39	−4193	−409912	−56040	−53	
$e^2\gamma^2$	−1	−42	−1614	− 849	+ 754	+68	+1
$e'^2\gamma^2$			+ 4	+ 1	+ 57		
γ^4		+ 1	+ 54	− 834	+ 48	+ 9	
$\gamma^2\alpha_1^2$					− 3		
$e^4\gamma^2$		− 6	+ 4	+ 11	+ 4		
$e^2\gamma^4$		+ 1	+ 4	− 31	+ 6	+ 1	
Sum ...	−1	−85	−5741	−411614	−55174	+25	+1

206. Arg. $l + 2i_1 D$.

$2i_1.$	5	3	1	−1	−3	−5
$e\alpha_1$	+1	+29	−8546	+18757	+3180	+ 5
$e^3\alpha_1$	−1	−38	− 127	+ 214	+ 108	+11
$ee'^2\alpha_1$			+ 6	+ 8	− 13	− 1
$e\gamma^2\alpha_1$		+ 7	+ 226	− 425	− 69	− 1
Sum ...		− 2	−8441	+18554	+3206	+14

207. Arg. $l' + 2i_1 D$.

$2i_1.$	5	3	1	−1	−3	−5
$e'\alpha_1$	+1	+112	+17654	+593	−90	−1
$e^2e'\alpha_1$	+1	+ 43	+ 599	+ 1	+27	
$e'^3\alpha_1$			− 3	+ 3		
$e'\gamma^2\alpha_1$		− 5	− 258	− 38	− 3	
Sum ...	+2	+150	+17992	+559	−66	−1

208. Arg. $3l + 2i D$.

$i.$	2	1	0	−1	−2	−3	−4
e^3	+17	+ 983	+36339	−13273	−1197	−296	−8
e^5	+ 5	+ 106	− 95	+ 13	+ 1	− 2	−1
$e^3e'^2$		− 3	− 8	− 5	− 2	+ 1	
$e^3\gamma^2$	− 1	− 26	− 112	+ 72	+ 11	+ 4	
Sum ...	+21	+1060	+36124	−13193	−1187	−293	−9

209. Arg. $2l+l'+2iD$.

$i.$	2	1	0	−1	−2	−3	−4
e^2e'	−5	−268	−7700	−8664	−2762	−85	−2
e^4e'	−2	− 33	− 45		− 6	− 8	−1
$e^2e'^3$			− 6		+ 2		
$e^2e'\gamma^2$		+ 11	+ 92	+ 26	+ 23	+ 2	
Sum ...	−7	−290	−7659	−8638	−2743	−91	−3

210. Arg. $2l-l'+2iD$.

$i.$	2	1	0	−1	−2	−3
e^2e'	+26	+1112	+9833	−2601	+352	+12
e^4e'	+ 8	+ 92	− 13	+ 30	+ 3	+ 2
$e^2e'^3$		− 1	+ 6	− 11	+ 1	
$e^2e'\gamma^2$	− 1	− 21	− 111	+ 85	+ 4	
Sum ...	+33	+1182	+9715	+2497	+360	+14

211. Arg. $l+2l'+2iD$.

$i.$	1	0	−1	−2	−3
ee'^2	− 1	−1170	−7445	−293	−6
$e^3e'^2$	−13	− 9	− 10	− 24	−2
$ee'^2\gamma^2$		+ 10	+ 24	+ 5	
Sum ...	−14	−1169	−7431	−312	−8

212. Arg. $l-2l'+2iD$.

$i.$	2	1	0	−1	−2
ee'^2	+19	+719	+2615	+2552	+18
$e^3e'^2$	+ 6	+ 49	+ 3	+ 4	+ 5
$ee'^2\gamma^2$	− 1	− 9	− 32	− 17	− 1
Sum ...	+24	+759	+2586	+2539	+22

213. Arg. $3l'+2i\mathrm{D}$.

i.	0	−1	−2
e'^3	− 98	−326	− 7
$e^2e'^3$	− 6	− 21	− 3
$e'^3\gamma^2$	+ 1	+ 2	
Sum ...	−103	−345	−10

214. Arg. $l+2\mathrm{F}+2i\mathrm{D}$.

i.	2	1	0	−1	−2	−3
$e\gamma^2$	−11	−809	−45068	−242	−309	−1
$e^3\gamma^2$	− 7	−196	+ 59	+ 63	+ 4	+1
$ee'^2\gamma^2$		+ 1	+ 1		+ 2	
$e\gamma^4$		+ 12	− 92		+ 2	
Sum ...	−18	−992	−45100	−179	−301	

215. Arg. $l-2\mathrm{F}+2i\mathrm{D}$.

i.	3	2	1	0	−1	−2	−3
$e\gamma^2$	−1	−55	−6331	+39316	+9367	+165	+2
$e^3\gamma^2$		−11	+ 10	− 239	+ 24	+ 40	+1
$ee'^2\gamma^2$			+ 10	− 22	− 4		
$e\gamma^4$		− 1	− 71	+ 477	− 21	− 3	
Sum ...	−1	−67	−6382	+39532	+9366	+202	+3

216. Arg. $l'+2\mathrm{F}+2i\mathrm{D}$.

i.	2	1	0	−1	−2
$e'\gamma^2$	+1	+47	+392	−2195	−15
$e^2e'\gamma^2$	+1	+20	+ 31	+ 36	+ 7
$e'^3\gamma^2$				+ 2	
$e'\gamma^4$		− 1	− 7	+ 2	+ 1
Sum ...	+2	+66	+416	−2155	− 7

217. Arg. $l'-2F+2iD$.

i.	1	0	−1	−2
$e'\gamma^2$	−1449	+59	+304	+ 6
$e^2e'\gamma^2$	+ 4	+24	+ 83	+ 5
$e'^3\gamma^2$				
$e'\gamma^4$	+ 3	− 7	− 3	
Sum...	−1442	+76	+384	+11

218. Arg. $2l+2i_1D$.

$2i_1$.	3	1	−1	−3	−5	−7
$e^2\alpha_1$	−2	−595	+1773	+1228	+57	+1
$e^4\alpha_1$	−3	− 6	+ 10	+ 3	+ 3	
$e^2\gamma^2\alpha_1$	+1	+ 17	− 38	− 10	− 1	
Sum...	−4	−584	+1745	+1221	+59	+1

219. Arg. $l+l'+2i_1D$.

$2i_1$.	3	1	−1	−3	−5
$ee'\alpha_1$	+19	+1244	+143	+230	
$e^3e'\alpha_1$	+ 5	+ 41	− 2	+ 7	+1
$ee'\gamma^2\alpha_1$	− 1	− 20	− 4	− 4	
Sum...	+23	+1265	+137	+233	+1

220. Arg. $l-l'+2i_1D$.

$2i_1$.	3	1	−1	−3	−5
$ee'\alpha_1$	+6	−122	−1062	−274	−4
$e^3e'\alpha_1$	−3	− 5	− 51	− 7	−1
$ee'\gamma^2\alpha_1$		+ 5	+ 26	+ 5	+2
Sum...	+3	−122	−1087	−276	−3

221. Arg. $2F+2i_1D$.

$2i_1$.	3	1	−1	−3	5
$\gamma^2\alpha_1$	+1	+254	+584	+258	+1
$e^2\gamma^2\alpha_1$	+3	+ 6		− 4	
$\gamma^4\alpha_1$		− 6	− 2	− 1	
Sum...	+4	+254	+582	+253	+1

222. Arg. $4l+2iD$.

i.	2	1	0	−1	−2	−3	−4
e^4	+2	+72	+1953	−957	+1	−14	−4
e^6			− 5				
$e^4\gamma^2$		− 2	− 10	+ 5	+2		
Sum...	+2	+70	+1938	−952	+3	−14	−4

223. Arg. $2l+2F+2iD$.

i.	2	1	0	−1	−2	−3
$e^2\gamma^2$	−2	−105	−4005	+558	−6	−3
$e^4\gamma^2$	−1	− 20	+ 15	+ 1	+1	
$e^2\gamma^4$		+ 2	− 6	− 2		
Sum...	−3	−123	−3996	+557	−5	−3

224. Arg. $2l-2F+2iD$.

i.	2	1	0	−1	−2	−3
$e^2\gamma^2$	−10	−450	−1352	+537	+171	+4
$e^4\gamma^2$	− 1	− 4	+ 80	− 3	+ 3	+1
$e^2\gamma^4$		− 5	− 26	+ 4	− 1	
Sum...	−11	−459	−1298	+538	+173	+5

225. Arg. $4F+2iD$.

i.	1	0	−1
γ^4	+ 8	+407	+77
$e^2\gamma^4$	+ 6	+ 11	− 3
Sum...	+14	+418	+74

226. Arg. $F+2iD$.

$i.$	3	2	1	0	−1	−2	−3
γ	+ 5	+ 633	+ 94476	+18517283	−618446	−2897	−19
γ^3		− 13	− 876	0	− 950	+ 3	
γe^2	+ 8	+ 527	+ 24010	− 55812	− 4857	− 793	−16
$\gamma e'^2$		− 2	− 92	0	+ 588	+ 11	
$\gamma \alpha_1{}^2$			+ 3	0	− 9		
γ^5			− 1	− 9	+ 4		
$\gamma^3 e^2$		− 9	− 121	0	− 2	+ 7	
$\gamma^3 e'^2$			+ 1	0			
γe^4	+ 2	+ 58	− 122	+ 18	+ 11	− 8	− 2
$\gamma e^2 e'^2$		− 2	− 16	0	+ 3	+ 2	
Sum ...	+15	+1192	−117262	+18461480	−623658	−3675	−37

227. Arg. $F+l+2iD$.

$i.$	3	2	1	0	−1	−2	−3	−4
γe	+1	+140	+13019	+1014212	−167571	−6536	−80	−1
$\gamma^3 e$		− 3	− 137	− 26	+ 590	+ 5		
γe^3	+1	+ 72	+ 2285	− 4001	+ 357	− 66	−15	
$\gamma e e'^2$		− 1	− 19	− 6	+ 49	+ 16		
$\gamma e \alpha_1{}^2$			+ 1		− 1			
γe^5		+ 6	− 14	+ 4	+ 1			
$\gamma^3 e^3$		− 1	− 13	− 6	− 2	+ 1		
$\gamma^5 e$				+ 3				
Sum ...	+2	+213	+15122	+1010180	−166577	−6580	−95	−1

228. Arg. $F-l+2iD$.

$i.$	3	2	1	0	−1	−2	−3
γe	+28	+2600	+201433	−997081	−33111	−401	−4
$\gamma^3 e$	− 1	− 41	− 997	− 3755	+ 213	+ 4	
γe^3	+14	+ 452	− 884	+ 1129	− 504	− 79	−2
$\gamma e e'^2$		− 5	− 73	+ 7	+ 44	+ 1	
$\gamma e \alpha_1{}^2$			+ 2		− 1		
γe^5	+ 1	− 2	+ 1	+ 2	+ 2	− 1	
$\gamma^3 e^3$		− 4	+ 3	+ 22	− 5	+ 1	
$\gamma^5 e$				− 19	+ 3		
Sum ...	+42	+3000	+199485	−999695	−33359	−475	−6

229. Arg. $F+l'+2iD$.

i	2	1	0	-1	-2	-3
$\gamma e'$	-13	-1002	-6125	-29443	-341	-4
$\gamma^3 e'$		$+19$	$+177$	-16		
$\gamma e^2 e'$	-11	-285	-550	-253	-77	-2
$\gamma e'^3$			$+6$	$+23$		
$\gamma e' \alpha_1^2$		-1				
Sum ...	-24	-1269	-6492	-29689	-418	-6

230. Arg. $F-l'+2iD$.

i.	3	2	1	0	-1	-2	-3
$\gamma e'$	$+1$	$+91$	$+6844$	$+4794$	$+12073$	$+89$	$+1$
$\gamma^3 e'$		-2	-52	-158	$+18$	-1	
$\gamma e^2 e'$	$+1$	$+65$	$+1215$	$+232$	$+49$	$+25$	$+1$
$\gamma e'^3$			-6	-5			
$\gamma e' \alpha_1^2$							
Sum ...	$+2$	$+154$	$+8001$	$+4863$	$+12140$	$+113$	$+2$

231. Arg. $F \quad 2i_1D$.

$2i_1$.	3	1	-1	-3	-5
$\gamma \alpha_1$	$+5$	-5418	$+4810$	$+320$	$+1$
$\gamma^3 \alpha_1$	$+2$	$+136$	-89	$+6$	
$\gamma e^2 \alpha_1$	-37	-74	$+73$	$+26$	$+2$
$\gamma e'^2 \alpha_1$	$+1$	-1	$+1$	-2	
Sum ...	-29	-5357	$+4795$	$+350$	$+3$

232. Arg. $3F+2iD$.

i.	2	1	0	-1	-2	-3
γ^3	-1	-92	-5978	-2277	-65	-1
γ^5		$+1$	-10	$+2$		
$\gamma^3 e^2$	-2	-52	-311	$+87$	$+2$	
$\gamma^3 e'^2$				$+3$		
Sum ...	-3	-143	-6299	-2185	-63	-1

233. Arg. $F+2l+2iD$.

$i.$	2	1	0	−1	−2	−3	−4
γe^2	+20	+1341	+62261	−15682	−638	−81	−2
$\gamma^3 e^2$		− 15	− 24	+ 65	+ 3	+ 1	
γe^4	+ 8	+ 200	− 319	+ 54		− 1	
$\gamma e^2 e'^2$		− 3	− 5	− 2			
Sum...	+28	+1523	+61913	−15565	−635	−81	−2

234. Arg. $F-2l+2iD$.

$i.$	4	3	2	1	0	−1	−2	−3
γe^2	+1	+53	+2451	−1630	−31504	−2136	−42	−1
$\gamma^3 e^2$		− 1	− 25	− 2	− 314	+ 30	+ 1	
γe^4		+ 8	− 10	+ 6	+ 46	− 44	− 7	
$\gamma e^2 e'^2$			− 3	+ 2	+ 9	+ 4		
Sum...	+1	+60	+2413	−1624	−31763	−2146	−48	−1

235. Arg. $F+l+l'+2iD$.

$i.$	2	1	0	−1	−2	−3
$\gamma e e'$	−4	−203	−5340	−7502	−593	−13
$\gamma^3 e e'$		+ 4	+ 61	+ 22		
$\gamma e^3 e'$	−2	− 40	− 49	+ 14	− 7	− 2
$\gamma e e'^3$			− 3	+ 3		
Sum ...	−6	−239	−5331	−7463	−600	−15

236. Arg. $F-l-l'+2iD$.

$i.$	3	2	1	0	−1	−2
$\gamma e e'$	+5	+303	+8975	+5118	+821	+14
$\gamma^3 e e'$		− 5	− 41	− 27	− 4	
$\gamma e^3 e'$	+2	+ 43	− 28	+ 1	+ 9	+ 3
$\gamma e e'^3$			− 4	+ 4		
Sum ...	+7	+341	+8902	+5096	+826	+17

237. Arg. $F+l-l'+2iD$.

$i.$	2	1	0	−1	−2	−3
$\gamma ee'$	+21	+1022	+6848	+769	+170	+3
$\gamma^3 ee'$		− 9	− 80	+ 17		
$\gamma e^3 e'$	+ 9	+ 129	− 16	+ 11	+ 1	
$\gamma ee'^3$		− 1	+ 4	− 2		
Sum ...	+30	+1141	+6756	+795	+171	+3

238. Arg. $F-l+l'+2iD$.

$i.$	3	2	1	0	−1	−2	−3
$\gamma ee'$	−1	−44	−1302	−5707	−1762	−50	−1
$\gamma^3 ee'$		+ 1	− 12	+ 42	+ 17	+ 1	
$\gamma e^3 e'$		− 8	− 11	+ 15	− 29	− 9	
$\gamma ee'^3$			+ 2	− 5	+ 1		
Sum ...	−1	−51	−1323	−5655	−1773	−58	−1

239. Arg. $F+2l'+2iD$.

$i.$	1	0	−1	−2
$\gamma e'^2$	− 1	−49	−1085	−25
$\gamma^3 e'^2$		+ 3	− 1	
$\gamma e^2 e'^2$	−15	−10	− 10	− 4
Sum ...	−16	−56	−1096	−29

240. Arg. $F-2l'+2iD$.

$i.$	2	1	0	−1	−2
$\gamma e'^2$	+ 8	+343	+16	+126	
$\gamma^3 e'^2$		− 3	− 1	+ 1	
$\gamma e^2 e'^2$	+ 5	+ 47	+ 4	+ 9	+1
Sum ...	+13	+387	+19	+136	+1

241. Arg. $F+l+2i_1D$.

$2i_1$.	3	1	−1	−3	−5
$\gamma e a_1$	−3	−678	+439	+306	+10
$\gamma^3 e a_1$	+1	+ 19	− 18	− 2	+ 1
$\gamma e^3 a_1$	−5	− 7	+ 8	+ 2	+ 1
Sum ...	−7	−666	+429	+306	+12

242. Arg. $F-l+2i_1D$.

$2i_1$.	5	3	1	−1	−3
$\gamma e a_1$	−1	−204	+136	+587	+33
$\gamma^3 e a_1$		+ 4	+ 6	+ 1	
$\gamma e^3 a_1$	−1	− 8	− 3	+ 3	+ 2
Sum ...	−2	−208	+139	+591	+35

243. Arg. $F+l'+2i_1D$.

$2i_1$.	3	1	−1	−3
$\gamma e' a$	+ 9	+795	+14	+23
$\gamma^3 e' a_1$		− 12	− 1	+ 1
$\gamma e^2 e' a_1$	+ 5	+ 21		+ 2
Sum...	+14	+804	+13	+26

244. Arg. $F-l'+2i_1D$.

$2i_1$.	3	1	−2	−3
$\gamma e' a_1$	+2	−20	−788	−32
$\gamma^3 e' a_1$		+ 2	+ 10	
$\gamma e^2 e' a_1$	−2		− 28	− 2
Sum...		−18	−806	−34

245. Arg. $3F+l+2iD$.

i.	1	0	−1	−2	−3
$\gamma^3 e$	−23	−992	−343	+6	−1
$\gamma^5 e$		−2	+1		
$\gamma^3 e^3$	−8	−27	+13	+1	
Sum ...	−31	−1021	−329	+7	−1

246. Arg. $3F-l+2iD$.

i.	2	1	0	−1	−2
$\gamma^3 e$	−5	−234	−2808	+290	+5
$\gamma^5 e$		+1	−39	+3	
$\gamma^3 e^3$	−2	−11	+33	−1	
Sum ...	−7	−244	−2814	+292	+5

247. Arg. $F+3l+2iD$.

i.	2	1	0	−1	−2	−3	−4
γe^3	+2	+124	+4015	−1528	+8	−7	−1
$\gamma^3 e^3$		−2	−4	+6			
γe^5	+1	+17	−27	+6	+1		
Sum ...	+3	+139	+3984	−1516	+9	−7	−1

248. Arg. $F-3l+2iD$.

i	4	3	2	1	0	−1	−2
γe^3	+1	+32	+22	+253	−1570	−146	−4
$\gamma^3 e^3$		−1	−1	+2	−19	+3	
γe^5					+4	−4	
Sum ...	+1	+31	+21	+255	−1585	−147	−4

249. Arg. $2iD$.

$i.$	3	2	1	0
1	+11	+1568	+245748	+34226987
e^2	+16	+ 985	+ 37988	0
e'^2		− 5	− 232	+ 11
k^2		− 12	− 924	0
$\alpha_1{}^2$			+ 4	
e^4	+ 5	+ 86	− 40	+ 1
$e^2e'^2$		− 5	− 25	0
$e^2\gamma^2$		− 10	− 187	+ 3
$e'^2\gamma^2$				0
γ^4			+ 1	− 1
Sum ..	+32	+2607	+282333	+34227001

250. Arg. $l+2iD$.

$i.$	3	2	1	0	−1	−2	−3	−4
e	+3	+305	+27534	+1866057	+345043	+5396	+61	+1
e^3	+3	+126	+ 3493	− 681	− 128	+ 678	+25	+1
ee'^2		− 2	− 39	− 13	− 118	− 11		
$e\gamma^2$		− 3	− 104	+ 32	− 1687	− 47	− 2	
$e\alpha_1{}^2$			+ 1		+ 4			
e^5	+1	+ 10	− 6			− 1	+ 2	
$e^3\gamma^2$		− 3	− 18	− 10		− 7		
$e\gamma^4$				+ 13	+ 3			
Sum...	+7	+433	+30861	+1865398	+343117	+6008	+86	+2

251. Arg. $l'+2iD$.

$i.$	2	1	0	−1	−2	−3
e'	−33	−2569	−3924	+17415	+223	+2
e^2e'	−20	− 451	− 127	+ 1874	+120	+4
e'^3			+ 2	− 14	− 2	
$e'\gamma^2$		+ 17	+ 47	− 73	− 2	
$e'\alpha_1{}^2$		− 1				
Sum ...	−53	−3004	−4002	+19202	+339	+6

252. Arg. $2i_1$D.

$2i_1$.	5	3	1
a_1	+1	+81	−9800
e^2a_1		−62	− 192
e'^2a_1		+ 1	− 2
γ^2a_1		+ 3	+ 242
Sum ...	+1	+23	−9752

253. Arg. $2l+2i$D.

i.	3	2	1	0	−1	−2	−3	−4
e^2		+41	+2546	+101788	−3052	+3760	+ 98	+1
e^4	+1	+13	+ 299	− 125	+ 1	+ 4	+ 11	+1
$e^2e'^2$			− 3	− 9		− 5		
$e^2\gamma^2$			− 9	+ 3	+ 12	− 37		
Sum...	+1	+54	+2833	+101657	−3039	+3722	+109	+2

254. Arg. $l+l'+2i$D.

i.	2	1	0	−1	−2	−3
ee'	− 8	−426	−9536	+14515	+619	+11
e^3e'	− 4	− 61	− 56	+ 7	+ 63	+ 4
ee'^3			− 7	− 5	− 2	
$ee'\gamma^2$		+ 2	+ 97	− 62	− 6	
Sum...	−12	−485	−9502	+14455	+674	+15

255. Arg. $l-l'+2i$D.

i.	3	2	1	0	−1	−2	−3
ee'	+1	+45	+2122	+11654	−2214	− 90	−3
e^3e'		+15	+ 195	+ 5	− 24	− 12	−2
ee'^3			− 4	+ 7	+ 4		
$ee'\gamma^2$			− 8	− 124	− 26		
Sum...	+1	+60	+2305	+11542	−2260	−102	−5

256. Arg. $2l' + 2i\mathrm{D}$.

i.	1	0	−1	−2
e'^2	− 2	−84	+853	+19
$e^2e'^2$	−26	− 4	+ 71	+ 9
$e^2\gamma^2$		+ 2	− 4	
Sum ...	−28	−86	+920	+28

257. Arg. $2\mathrm{F} + 2i\mathrm{D}$.

i.	1	0	−1	−2	−3
γ^2	+ 9	+667	−1071	+41	+1
$e^2\gamma^2$	−17	−788	+ 16	− 9	
$e'^2\gamma^2$			+ 2		
γ^4	− 1	− 3	+ 1	− 1	
Sum ...	− 9	−124	−1052	+31	+1

258. Arg. $l + 2i_1\mathrm{D}$.

$2i_1$.	3	1	−1	−3	−5
ea_1	+2	−1103	+120	−378	−1
e^3a_1	−7	− 16	+ 2	− 15	−2
$e\gamma^2a_1$	+2	+ 29	− 4	+ 8	
Sum ...	−3	−1090	+118	−385	−3

259. Arg. $l' + 2i_1\mathrm{D}$.

$2i$.	3	1	−1	−3
$e'a_1$	+20	+1464	−40	+10
$e^2e'a_1$	+ 7	+ 50		− 3
$e'\gamma^2a_1$		− 22	+ 3	
Sum ...	+27	+1492	−37	+ 7

260. Arg. $3l+2iD$.

$i.$	2	1	0	−1	−2	−3	−4
e^3	+5	+219	+6231	−1192	+76	+47	+2
e^5	+2	+ 24	− 15	+ 1		+ 1	
$e^3\gamma^2$			− 1	+ 4	− 2	− 2	
Sum ...	+7	+243	+6215	−1187	+74	+46	+2

261. Arg. $l+2F+2iD$.

$i.$	1	0	−1	−2	−3
$e\gamma^2$	+1	+63	−847	+15	+2
$e^3\gamma^2$		−73	+ 11	− 1	
$e\gamma^4$			+ 3		
Sum ...	+1	−10	−833	+14	+2

262. Arg. $l-2F+2iD$.

$i.$	2	1	0	−1	−2
$e\gamma^2$	−2	−476	−7063	− 88	−1
$e^3\gamma^2$	−3	− 1	+ 29	− 24	
$e\gamma^4$		− 4	− 102		
Sum ...	−5	−481	−7136	−112	−1

Section (v). *The Final Values of the Coefficients in Longitude, Latitude, and Parallax.*

263. The following tables, giving the final values of the coefficients, are arranged, first, according to the order and composition of the principal characteristics; second, according to the signs of the multiples of l, l', F; and third, according to multiples of D in descending order.

In the first column, headed "P. C.," is given the principal characteristic. In the second, third, fourth, and fifth columns, headed "l," "l'," "F," "D," are given the multiples of those arguments (DELAUNAY'S notation) to which the coefficients in the last column correspond. The characteristic is understood to belong to all coefficients down to the next printed characteristic; a similar remark applies to the multiples of l, l', F.

As stated earlier, the system of axes used is the same as that in which DELAUNAY expressed his final results.

These coefficients are all definitive for the corresponding arguments, with the following exceptions:—

(α) Small changes due to possible changes in the values of the arbitrary constants.

(β) Small additions due to the terms arising from the perturbations noted in (a), (b), (c) of § 4, Chap. I. These are very minute and are easily obtained. They are, however, more simply treated by the method of the variation of arbitrary constants, and will therefore be given with the treatment of the planetary inequalities.

The results have been discussed, and a comparison has been made with those of HANSEN elsewhere.*

* E. W. Brown, "The Final Values of the Coefficients in the New Lunar Theory," *Monthly Notices*, January 1905.

264. *Longitude.* Coefficients of Sines.

P.C.	l.	l'.	F.	D.	Coeff.
1	0	0	0	8	+ 0″·001
				6	+ ·127
				4	+ 13·902
				2	+ 2369·899
e	1	0	0	6	+ ·023
				4	+ 1·979
				2	+ 191·954
				0	+ 22639·580
				−2	− 4586·438
				−4	− 38·428
				−6	− ·393
				−8	− ·004
e'	0	1	0	6	− ·004
				4	− ·289
				2	− 24·451
				0	− 668·944
				−2	− 165·351
				−4	− 1·879
				−6	− ·024
a_1	0	0	0	5	+ ·004
				3	+ ·402
				1	− 124·785
e^2	2	0	0	6	+ ·004
				4	+ ·213
				2	+ 14·387
				0	+ 769·021
				−2	− 211·658
				−4	− 30·773
				−6	− ·570
				−8	− ·009
ee'	1	1	0	4	− ·051
				2	− 2·925
				0	− 109·804
				−2	− 206·219
				−4	− 4·396
				−6	− ·072
				−8	− ·001

P.C.	l.	l'.	F.	D.	Coeff.
ee'	1	−1	0	6	+ 0″·005
				4	+ ·283
				2	+ 14·595
				0	+ 147·878
				−2	+ 28·511
				−4	+ ·637
				−6	+ ·011
e'^2	0	2	0	4	− ·003
				2	− ·189
				0	− 7·505
				−2	− 8·116
				−4	− ·151
				−6	− ·002
γ^2	0	0	2	6	− ·001
				4	− ·085
				2	− 5·741
				0	− 411·614
				−2	− 55·174
				−4	+ ·025
				−6	+ ·001
$e'a_1$	1	0	0	3	− ·002
				1	− 8·441
				−1	+ 18·554
				−3	+ 3·206
				−5	+ ·014
$e'a_1$	0	1	0	5	+ ·002
				3	+ ·150
				1	+ 17·992
				−1	+ ·559
				−3	− ·066
				−5	− ·001

Longitude. Coefficients of Sines (*continued*).

P. C.	*l.*	*l′.*	F.	D.	Coeff.
e^3	3	0	0	4	+ 0″·021
				2	+ 1·060
				0	+36·124
				−2	−13·193
				−4	− 1·187
				−6	− ·293
				−8	− ·009
e^2e'	2	1	0	4	− ·007
				2	− ·290
				0	− 7·659
				−2	− 8·638
				−4	− 2·743
				−6	− ·091
				−8	− ·003
	2	−1	0	4	+ ·033
				2	+ 1·182
				0	+ 9·715
				−2	− 2·497
				−4	+ ·360
				−6	+ ·014
ee'^2	1	2	0	2	− ·014
				0	− 1·169
				−2	− 7·431
				−4	− ·312
				−6	− ·008
	1	−2	0	4	+ ·024
				2	+ ·759
				0	+ 2·586
				−2	+ 2·539
				−4	+ ·022
e'^3	0	3	0	0	− ·103
				−2	− ·345
				−4	− ·010

P. C.	*l.*	*l′.*	F.	D.	Coeff.
$e\gamma^2$	1	0	2	4	− 0″·018
				2	− ·992
				0	−45·100
				−2	− ·179
				−4	− ·301
	1	0	−2	6	− ·001
				4	− ·067
				2	− 6·382
				0	+39·532
				−2	+ 9·366
				−4	+ ·202
				−6	+ ·003
$e'\gamma^2$	0	1	2	4	+ ·002
				2	+ ·066
				0	+ ·416
				−2	− 2·155
				−4	− ·007
	0	1	−2	2	− 1·442
				0	+ ·076
				−2	+ ·384
				−4	+ ·011
$e^2\alpha_1$	2	0	0	3	− ·004
				1	− ·584
				−1	+ 1·745
				−3	+ 1·221
				−5	+ ·059
				−7	+ ·001
$ee'\alpha_1$	1	1	0	3	+ ·023
				1	+ 1·265
				−1	+ ·137
				−3	+ ·233
				−5	+ ·001

Longitude. Coefficients of Sines (*continued*).

P. C.	*l.*	*l'.*	F.	D.	Coeff.
$ee'\alpha_1$	1	−1	0	3	+0″·003
				1	− ·122
				−1	−1·087
				−3	− ·276
				−5	− ·003
$e'^2\alpha_1$	0	2	0	3	− ·002
				1	− ·039
				−1	− ·042
				−3	− ·006
$\gamma^2\alpha_1$	0	0	2	3	+ ·004
				1	+ ·254
				−1	+ ·582
				−3	+ ·253
				−5	+ ·001
e^4	4	0	0	4	+ ·002
				2	+ ·070
				0	+1·938
				−2	− ·952
				−4	+ ·003
				−6	− ·014
				−8	− ·004
e^3e'	3	1	0	2	− ·025
				0	− ·552
				−2	− ·483
				−4	− ·100
				−6	− ·039
				−8	− ·001
	3	−1	0	4	+ ·003
				2	+ ·088
				0	+ ·682
				−2	− ·183
				−4	− ·029
				−6	+ ·005

P. C.	*l.*	*l'.*	F.	D.	Coeff.
$e^2e'^2$	2	2	0	0	−0″·067
				−2	− ·298
				−4	− ·161
				−6	− ·008
	2	−2	0	4	+ ·003
				2	+ ·062
				0	+ ·197
				−2	+ ·255
				−4	+ ·036
				−6	+ ·001
ee'^3	1	3	0	0	− ·018
				−2	− ·250
				−4	− ·016
	1	−3	0	4	+ ·001
				2	+ ·032
				0	+ ·051
				−2	+ ·003
e'^4	0	4	0	0	− ·001
				−2	− ·013
$e^2\gamma^2$	2	0	2	4	− ·003
				2	− ·123
				0	−3·996
				−2	+ ·557
				−4	− ·005
				−6	− ·003
	2	0	−2	4	− ·011
				2	− ·459
				0	−1·298
				−2	+ ·538
				−4	+ ·173
				−6	+ ·005

Longitude. Coefficients of Sines (*continued*).

P. C.	*l*.	*l'*.	F.	D.	Coeff.
$ee'\gamma^2$	1	1	2	2	+0″·012
				0	+ ·263
				−2	+ ·059
				−4	− ·024
				−6	− ·001
	1	1	−2	4	+ ·002
				2	+ ·083
				0	− ·083
				−2	+ ·427
				−4	+ ·019
	1	−1	2	4	− ·002
				2	− ·064
				0	− ·304
				−2	+ ·002
				−4	+ ·018
	1	−1	−2	4	− ·007
				2	− ·372
				0	+ ·083
				−2	− ·065
				−4	− ·002
$e'^2\gamma^2$	0	2	2	0	+ ·004
				−2	− ·066
				−4	− ·002
	0	2	−2	2	− ·025
				0	− ·002
				−2	+ ·016
γ^4	0	0	4	2	+ ·014
				0	+ ·418
				−2	+ ·074
$e^3\alpha_1$	3	0	0	1	− ·042
				−1	+ ·130
				−3	+ ·045
				−5	+ ·016
				−7	+ ·001

P. C.	*l*.	*l'*.	F.	D.	Coeff.
$e^2e'\alpha_1$	2	1	0	3	+0″·003
				1	+ ·092
				−1	+ ·006
				−3	+ ·084
				−5	+ ·006
	2	−1	0	1	− ·014
				−1	− ·352
				−3	+ ·042
				−5	− ·003
$ee'^2\alpha_1$	1	2	0	1	− ·008
				−1	− ·002
				−3	+ ·012
	1	−2	0	−1	+ ·003
				−3	+ ·001
$e'^3\alpha_1$	0	3	0	1	− ·001
				−1	− ·002
$e\gamma^2\alpha_1$	1	0	2	1	+ ·045
				−1	+ ·024
				−3	+ ·030
				−5	+ ·002
	1	0	−2	3	− ·010
				1	− ·041
				−1	− ·016
				−3	− ·011
$e'\gamma^2\alpha_1$	0	1	2	3	− ·001
				1	− ·035
				−1	+ ·013
				−3	+ ·020
	0	1	−2	3	+ ·009
				1	− ·001
				−1	− ·002

Longitude. Coefficients of Sines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
e^5	5	0	0	2	$+0''.005$
				0	$+$ ·113
				-2	$-$ ·069
				-4	$+$ ·004
e^4e'	4	1	0	2	$-$ ·002
				0	$-$ ·040
				-2	$-$ ·030
				-4	·000
				-6	$-$ ·002
				-8	$-$ ·001
	4	-1	0	2	$+$ ·007
				0	$+$ ·048
				-2	$-$ ·019
				-4	$-$ ·001
$e^3e'^2$	3	2	0	0	$-$ ·003
				-2	$-$ ·016
				-4	$-$ ·006
				-6	$-$ ·003
	3	-2	0	2	$+$ ·005
				0	$+$ ·016
				-2	$+$ ·011
				-4	$+$ ·004
				-6	$+$ ·001
$e^2e'^3$	2	3	0	0	$-$ ·001
				-2	$-$ ·010
				-4	$-$ ·008
				-6	$-$ ·001
	2	-3	0	2	$+$ ·003
				0	$+$ ·004
				-2	$+$ ·001
$e^3\gamma^2$	3	0	2	4	$-$ ·003
				2	$-$ ·011
				0	$-$ ·330
				-2	$+$ ·092

P. C.	l.	l'.	F.	D.	Coeff.
$e^3\gamma^2$	3	0	-2	4	$-0''.001$
				2	$-$ ·033
				0	$-$ ·055
				-2	$-$ ·005
				-4	$+$ ·009
				-6	$+$ ·003
$e^2e'\gamma^2$	2	1	2	2	$+$ ·002
				0	$+$ ·043
				-2	$+$ ·028
	2	1	-2	2	$+$ ·009
				0	$+$ ·026
				-2	$+$ ·022
				-4	$+$ ·016
				-6	$+$ ·001
	2	-1	2	2	$-$ ·009
				0	$-$ ·053
				-2	$+$ ·004
	2	-1	-2	4	$-$ ·001
				2	$-$ ·029
				0	$-$ ·024
				-2	·000
				-4	$-$ ·002
$ee'^2\gamma^2$	1	2	2	0	$+$ ·003
				-2	$+$ ·004
				-4	$-$ ·001
	1	2	-2	2	$-$ ·002
				0	·000
				-2	$+$ ·015
				-4	$+$ ·001
	1	-2	2	2	$-$ ·003
				0	$-$ ·005
				-2	$+$ ·007
				-4	$-$ ·001

Longitude. Coefficients of Sines (*concluded*).

P. C.	l.	l'.	F.	D.	Coeff.
$ee'^2\gamma^2$	1	−2	−2	4	−0″·001
				2	− ·016
				0	·000
				−2	− ·005
$e'^3\gamma^2$	0	3	2	−2	− ·002
	0	3	−2	−2	+ ·001
$e\gamma^4$	1	0	4	2	+ ·003
				0	+ ·090
				−2	+ ·009
	1	0	−4	4	− ·001
				2	+ ·001
				0	− ·080
				−2	− ·019
$e'\gamma^4$	0	1	4	0	− ·001
				−2	+ ·003
	0	1	−4	2	+ ·002
				0	·000
				−2	− ·001
e^4a_1	4	0	0	1	− ·003
				−1	+ ·010
				−3	+ ·002
				−5	+ ·001
$e^3e'a_1$	3	1	0	1	+ ·007
				−1	− ·001
				−3	+ ·003
				−5	+ ·002
	3	−1	0	1	− ·002
				−1	− ·023
				−3	+ ·007
$e^2\gamma^2a_1$	2	0	2	1	+ ·006
				−1	− ·003

P. C.	l.	l'.	F.	D.	Coeff.
$e^2\gamma^2a_1$	2	0	−2	3	−0″·001
				1	− ·001
				−1	+ ·001
				−3	− ·003
$ee'\gamma^2a_1$	1	1	2	1	− ·006
				−1	+ ·001
				−3	+ ·002
	1	1	−2	1	− ·002
				−1	·000
				−3	− ·001
	1	−1	2	−3	− ·001
	1	−1	−2	3	− ·001
				1	− ·004
				−1	·000
				−3	+ ·001
γ^4a_1	0	0	4	1	− ·001
				−1	− ·001
e^6	6	0	0	0	+ ·007
$e^4\gamma^2$	4	0	2	2	− ·001
				0	− ·025
				−2	+ ·010
				−4	− ·001
	4	0	−2	2	− ·001
				0	− ·007
				−2	+ ·002
$e^2\gamma^4$	2	0	4	2	+ ·001
				0	+ ·011
	2	0	−4	0	+ ·001
				−2	− ·003
				−4	− ·001

265. *Latitude.* Coefficients of Sines.

P. C.	$l.$	$l'.$	F.	D.	Coeff.
γ	0	0	1	6	+ 0″·015
				4	+ 1·192
				2	+ 117·262
				0	+ 18461·480
				−2	− 623·658
				−4	− 3·675
				−6	− ·037
γe	1	0	1	6	+ ·002
				4	+ ·213
				2	+ 15·122
				0	+ 1010·180
				−2	− 166·577
				−4	− 6·580
				−6	− ·095
				−8	− ·001
	−1	0	1	6	+ ·042
				4	+ 3·000
				2	+ 199·485
				0	− 999·695
				−2	− 33·359
				−4	− ·475
				−6	− ·006
$\gamma e'$	0	1	1	4	− ·024
				2	− 1·269
				0	− 6·492
				−2	− 29·689
				−4	− ·418
				−6	− ·006
	0	−1	1	6	+ ·002
				4	+ ·154
				2	+ 8·001
				0	+ 4·863
				−2	+ 12·140
				−4	+ ·113
				−6	+ ·002

P. C.	$l.$	$l'.$	F.	D.	Coeff.
γa_1	0	0	1	3	− 0″·029
				1	− 5·357
				−1	+ 4·795
				−3	+ ·350
				−5	+ ·003
γ^3	0	0	3	4	− ·003
				2	− ·143
				0	− 6·299
				−2	− 2·185
				−4	− ·063
				−6	− ·001
γe^2	2	0	1	4	+ ·028
				2	+ 1·523
				0	+ 61·913
				−2	− 15·565
				−4	− ·635
				−6	− ·081
				−8	− ·002
	−2	0	1	8	+ ·001
				6	+ ·060
				4	+ 2·413
				2	− 1·624
				0	− 31·763
				−2	− 2·146
				−4	− ·048
				−6	− ·001
$\gamma ee'$	1	1	1	4	− ·006
				2	− ·239
				0	− 5·331
				−2	− 7·463
				−4	− ·600
				−6	− ·015

Latitude. Coefficients of Sines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
$\gamma e e'$	−1	−1	1	6	+0″·007
				4	+ ·341
				2	+8·902
				0	+5·096
				−2	+ ·826
				−4	+ ·017
	1	−1	1	4	+ ·030
				2	+1·141
				0	+6·756
				−2	+ ·795
				−4	+ ·171
				−6	+ ·003
	−1	1	1	6	− ·001
				4	− ·051
				2	−1·323
				0	−5·655
				−2	−1·773
				−4	− ·058
				−6	− ·001
$\gamma e'^2$	0	2	1	2	− ·016
				0	− ·056
				−2	−1·096
				−4	− ·029
	0	−2	1	4	+ ·013
				2	+ ·387
				0	+ ·019
				−2	+ ·136
				−4	+ ·001
$\gamma e a_1$	1	0	1	3	− ·007
				1	− ·666
				−1	+ ·429
				−3	+ ·306
				−5	+ ·012
$\gamma e a_1$	−1	0	1	5	−0″·002
				3	− ·208
				1	+ ·139
				−1	+ ·591
				−3	+ ·035
$\gamma e' a_1$	0	1	1	3	+ ·014
				1	+ ·804
				−1	+ ·013
				−3	+ ·026
	0	−1	1	1	− ·018
				−1	− ·806
				−3	− ·034
$\gamma^3 e$	1	0	3	2	− ·031
				0	−1·021
				−2	− ·329
				−4	+ ·007
				−6	− ·001
	−1	0	3	4	− ·007
				2	− ·244
				0	−2·814
				−2	+ ·292
				−4	+ ·005
$\gamma^3 e'$	0	1	3	0	+ ·011
				−2	− ·093
				−4	− ·006
	0	−1	3	2	− ·007
				0	+ ·001
				−2	+ ·056
				−4	+ ·003

Latitude. Coefficients of Sines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
γe^3	3	0	1	4	$+0''.003$
				2	+ ·139
				0	+3·984
				−2	−1·516
				−4	+ ·009
				−6	− ·007
				−8	− ·001
	−3	0	1	8	+ ·001
				6	+ ·031
				4	+ ·021
				2	+ ·255
				0	−1·585
				−2	− ·147
				−4	− ·004
$\gamma e^2 e'$	2	1	1	2	− ·027
				0	− ·644
				−2	− ·657
				−4	− ·053
				−6	− ·011
	−2	−1	1	6	+ ·009
				4	+ ·217
				2	− ·063
				0	+ ·314
				−2	+ ·063
				−4	+ ·001
	2	−1	1	4	+ ·003
				2	+ ·114
				0	+ ·809
				−2	− ·084
				−4	+ ·002
				−6	+ ·002
$\gamma e^2 e'$	−2	1	1	6	$-0''.001$
				4	− ·029
				2	+ ·056
				0	− ·303
				−2	− ·129
				−4	− ·005
$\gamma e e'^2$	1	2	1	0	− ·055
				−2	− ·272
				−4	− ·034
				−6	− ·001
	−1	−2	1	6	+ ·001
				4	+ ·022
				2	+ ·319
				0	+ ·062
				−2	+ ·006
	1	−2	1	4	+ ·002
				2	+ ·054
				0	+ ·117
				−2	+ ·107
				−4	+ ·004
	−1	2	1	4	− ·001
				2	− ·115
				0	− ·096
				−2	− ·069
				−4	− ·003
$\gamma e'^3$	0	3	1	−2	− ·037
				−4	− ·001
	0	−3	1	2	+ ·014
$\gamma^3 a_1$	0	0	3	1	+ ·006
				−1	+ ·032
				−3	+ ·010

Latitude. Coefficients of Sines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
$\gamma e^2 \alpha_1$	2	0	1	3	−0″·001
				1	− ·065
				−1	+ ·112
				−3	+ ·039
				−5	+ ·005
	−2	0	1	5	− ·005
				3	− ·049
				1	− ·078
				−1	+ ·036
				−3	+ ·003
$\gamma e e' \alpha_1$	1	1	1	3	+ ·002
				1	+ ·101
				−1	− ·011
				−3	+ ·021
				−5	+ ·001
	−1	−1	1	3	− ·013
				1	+ ·001
				−1	− ·034
				−3	− ·004
	1	−1	1	1	− ·009
				−1	− ·006
				−3	− ·013
				−5	− ·001
	−1	1	1	3	+ ·022
				1	− ·056
				−1	+ ·020
				−3	+ ·002
$\gamma e'^2 \alpha_1$	0	2	1	1	− ·002
				−1	− ·002
				−3	+ ·001
	0	−2	1	1	+ ·002
				−1	+ ·002

P. C.	l.	l'.	F.	D.	Coeff.
γ^5	0	0	5	0	+0″·005
				−2	+ ·002
$\gamma^3 e^2$	2	0	3	2	− ·004
				0	− ·116
				−2	− ·022
				−4	+ ·005
	−2	0	3	4	− ·006
				2	− ·066
				0	+ ·130
				−2	+ ·010
				−4	+ ·001
$\gamma^3 e e'$	1	1	3	0	+ ·007
				−2	− ·011
	−1	−1	3	2	− ·011
				0	+ ·008
				−2	− ·005
	1	−1	3	2	− ·002
				0	− ·006
				−2	+ ·003
	−1	1	3	2	+ ·002
				0	− ·009
				−2	+ ·017
$\gamma^3 e'^2$	0	2	3	−2	− ·003
				−4	− ·001
	0	−2	3	−2	+ ·001
γe^4	4	0	1	2	+ ·011
				0	+ ·266
				−2	− ·135
				−4	+ ·007
	−4	0	1	6	+ ·001
				4	·000
				2	+ ·025
				0	− ·091
				−2	− ·010

Latitude. Coefficients of Sines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
$\gamma e^3 e'$	3	1	1	2	$-0''.003$
				0	$-$ ·063
				-2	$-$ ·056
				-4	+ ·001
				-6	$-$ ·001
	-3	-1	1	6	+ ·004
				4	+ ·002
				2	+ ·010
				0	+ ·024
				-2	+ ·005
	3	-1	1	2	+ ·011
				0	+ ·076
				-2	$-$ ·019
	-3	1	1	6	$-$ ·001
				4	+ ·002
				2	+ ·002
				0	$-$ ·024
				-2	$-$ ·010
$\gamma e^2 e'^2$	2	2	1	0	$-$ ·005
				-2	$-$ ·023
				-4	$-$ ·003
				-6	$-$ ·001
	-2	-2	1	6	+ ·001
				4	+ ·013
				2	$-$ ·002
				0	+ ·003
	2	-2	1	2	+ ·007
				0	+ ·016
				-2	+ ·013
				-4	+ ·001
	-2	2	1	4	$-$ ·003
				2	$-$ ·006
				0	$-$ ·006
				-2	$-$ ·005

P. C.	l.	l'.	F.	D	Coeff.
$\gamma e e'^3$	1	3	1	0	$-0''.001$
				-2	$-$ ·010
				-4	$-$ ·001
	-1	-3	1	4	+ ·001
				2	+ ·011
				0	+ ·001
	1	-3	1	2	+ ·002
				0	+ ·002
	-1	3	1	0	$-$ ·002
				-2	$-$ ·002
$\gamma^3 e a_1$	1	0	3	1	+ ·002
				-1	+ ·004
				-3	+ ·002
				-5	$-$ ·001
	-1	0	3	3	+ ·001
				1	+ ·003
				-1	$-$ ·001
				-3	$-$ ·001
$\gamma^3 e' a_1$	0	1	3	1	$-$ ·001
				-1	·000
				-3	+ ·001
	0	-1	3	-1	$-$ ·001
$\gamma e^3 a_1$	3	0	1	1	$-$ ·006
				-1	+ ·013
				-3	+ ·003
	-3	0	1	5	$-$ ·002
				3	·000
				1	$-$ ·005
				-1	+ ·002
$\gamma e^2 e' a_1$	2	1	1	1	+ ·010
				-1	$-$ ·001
				-3	+ ·003

Latitude. Coefficients of Sines (*concluded*).

P. C.	l.	l'.	F.	D.	Coeff.
$\gamma e^2 e' a_1$	−2	−1	1	3	−0″·003
				1	− ·002
				−1	− ·003
	2	−1	1	1	− ·002
				−1	− ·016
				−3	+ ·001
	−2	1	1	3	− ·003
				1	+ ·017
				−1	+ ·001
$\gamma^5 e$	1	0	5	0	+ ·002
				−2	+ ·001
	−1	0	5	0	+ ·003
				−2	+ ·001

P. C.	l.	l'.	F.	D.	Coeff.
$\gamma^3 e^3$	3	0	3	2	−0″·001
				0	− ·014
	−3	0	3	4	− ·003
				2	+ ·002
				0	+ ·001
				−2	+ ·001
γe^5	5	0	1	2	+ ·001
				0	+ ·018
				−2	− ·012
				−4	+ ·001
	−5	0	1	2	+ ·002
				0	− ·006
				−2	− ·001

266. *Parallax.* Coefficients of Cosines.

P. C.	l.	l'.	F.	D.	Coeff.
I	0	0	0	6	+ 0''·0032
				4	+ ·2607
				2	+ 28·2333
				0	+3422·7000
e	1	0	0	6	+ ·0007
				4	+ ·0433
				2	+ 3·0861
				0	+ 186·5398
				−2	+ 34·3117
				−4	+ ·6008
				−6	+ ·0086
				−8	+ ·0002
e'	0	1	0	4	− ·0053
				2	− ·3004
				0	− ·4002
				−2	+ 1·9202
				−4	+ ·0339
				−6	+ ·0006
a_1	0	0	0	5	+ ·0001
				3	+ ·0023
				1	− ·9752
e^2	2	0	0	6	+ ·0001
				4	+ ·0054
				2	+ ·2833
				0	+ 10·1657
				−2	− ·3039
				−4	+ ·3722
				−6	+ ·0109
				−8	+ ·0002

P. C.	l.	l'.	F.	D.	Coeff.
ee'	1	1	0	4	−0''·0012
				2	− ·0485
				0	− ·9502
				−2	+1·4455
				−4	+ ·0674
				−6	+ ·0015
	1	−1	0	6	+ ·0001
				4	+ ·0060
				2	+ ·2305
				0	+1·1542
				−2	− ·2260
				−4	− ·0102
				−6	− ·0005
e'^2	0	2	0	2	− ·0028
				0	− ·0086
				−2	+ ·0920
				−4	+ ·0028
γ^2	0	0	2	2	− ·0009
				0	− ·0124
				−2	− ·1052
				−4	+ ·0031
				−6	+ ·0001
ea_1	1	0	0	3	− ·0003
				1	− ·1090
				−1	+ ·0118
				−3	− ·0385
				−5	− ·0003
$e'a_1$	0	1	0	3	+ ·0027
				1	+ ·1492
				−1	− ·0037
				−3	+ ·0007

Parallax. Coefficients of Cosines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
e^3	3	0	0	4	+0″·0007
				2	+ ·0243
				0	+ ·6215
				−2	− ·1187
				−4	+ ·0074
				−6	+ ·0046
				−8	+ ·0002
e^2e'	2	1	0	4	− ·0001
				2	− ·0051
				0	− ·1039
				−2	− ·0192
				−4	+ ·0324
				−6	+ ·0017
	2	−1	0	4	+ ·0007
				2	+ ·0213
				0	+ ·1270
				−2	− ·0017
				−4	− ·0043
				−6	− ·0002
ee'^2	1	2	0	2	+ ·0001
				0	− ·0106
				−2	+ ·0485
				−4	+ ·0044
				−6	+ ·0002
	1	−2	0	4	+ ·0005
				2	+ ·0112
				0	+ ·0196
				−2	− ·0213
				−4	− ·0003
e'^3	0	3	0	0	− ·0002
				−2	+ ·0036
				−4	+ ·0002

P. C.	l.	l'.	F.	D.	Coeff.
$e\gamma^2$	1	0	2	2	+0″·0001
				0	− ·0010
				−2	− ·0833
				−4	+ ·0014
				−6	+ ·0002
	1	0	−2	4	− ·0005
				2	− ·0481
				0	− ·7136
				−2	− ·0112
				−4	− ·0001
$e'\gamma^2$	0	1	2	0	+ ·0013
				−2	− ·0066
				−4	+ ·0005
	0	1	−2	4	− ·0001
				2	+ ·0014
				0	+ ·0017
				−2	+ ·0001
$e^2\alpha_1$	2	0	0	1	− ·0100
				−1	+ ·0155
				−3	− ·0088
				−5	− ·0008
$ee'\alpha_1$	1	1	0	3	+ ·0003
				1	+ ·0164
				−1	·0000
				−3	− ·0025
	1	−1	0	1	− ·0014
				−1	·0000
				−3	+ ·0036
$e'^2\alpha_1$	0	2	0	1	− ·0003
				−1	+ ·0003
				−3	+ ·0001
$\gamma^2\alpha_1$	0	0	2	1	+ ·0001
				−1	+ ·0071
				−3	− ·0017

Parallax. Coefficients of Cosines (*continued*).

P. C.	l.	l'.	F.	D.	Coeff.
e^4	4	0	0	4	+0″·0001
				2	+ ·0018
				0	+ ·0401
				−2	− ·0130
				−4	+ ·0001
				−6	+ ·0002
				−8	+ ·0001
e^3e'	3	1	0	2	− ·0006
				0	− ·0097
				−2	− ·0045
				−4	+ ·0006
				−6	+ ·0005
				−8	+ ·0001
	3	−1	0	4	+ ·0001
				2	+ ·0017
				0	+ ·0115
				−2	− ·0017
				−4	+ ·0002
				−6	− ·0001
$e^2e'^2$	2	2	0	0	− ·0009
				−2	− ·0009
				−4	+ ·0020
				−6	+ ·0001
	2	−2	0	4	+ ·0001
				2	+ ·0013
				0	+ ·0024
				−2	− ·0001
				−4	− ·0005
ee'^3	1	3	0	0	− ·0002
				−2	+ ·0014
				−4	+ ·0002
	1	−3	0	2	+ ·0004
				0	+ ·0004
				−2	+ ·0002

P. C.	l.	l'.	F.	D.	Coeff.
$e^2\gamma^2$	2	0	2	2	−0″·0001
				0	+ ·0004
				−2	− ·0090
				−4	+ ·0002
	2	0	−2	2	− ·0053
				0	+ ·0004
				−2	− ·0141
				−4	− ·0004
$ee'\gamma^2$	1	1	2	0	+ ·0001
				−2	− ·0032
				−4	+ ·0001
	1	1	−2	2	+ ·0006
				0	+ ·0024
				−2	− ·0006
	1	−1	2	2	− ·0001
				0	+ ·0003
				−2	+ ·0004
	1	−1	−2	4	− ·0001
				2	− ·0027
				0	− ·0029
$e'^2\gamma^2$	0	2	2	−2	− ·0004
γ^4	0	0	4		·0000
e^3a_1	3	0	0	1	− ·0009
				−1	+ ·0017
				−3	+ ·0001
				−5	− ·0002
$e^2e'a_1$	2	1	0	3	+ ·0002
				1	+ ·0015
				−1	− ·0002
				−3	− ·0005
				−5	− ·0002
	2	−1	0	1	− ·0005
				−1	− ·0028
				−3	− ·0005
				−5	+ ·0002

Parallax. Coefficients of Cosines (*concluded*).

P. C.	l.	l'.	F.	D.	Coeff.
$e\gamma^2\alpha_1$	1	0	2	1	+0″·0002
				−1	+ ·0010
				−3	+ ·0002
				−5	− ·0002
	1	0	−2	3	− ·0002
				1	·0000
				−1	+ ·0006
				−3	+ ·0004
$e'\gamma^2\alpha_1$	0	1	2	−1	+ ·0001
				−3	− ·0001
	0	1	−2	3	+ ·0001
				1	− ·0003

P.C.	l.	l'.	F.	D.	Coeff.
e^5	5	0	0	2	+0″·0002
				0	+ ·0026
				−2	− ·0012
				−4	+ ·0001
$e^3\gamma^2$	3	0	2	2	− ·0001
				0	·0000
				−2	− ·0009
				−4	+ ·0001
	3	0	−2	2	− ·0005
				0	− ·0003
				−2	+ ·0001
				−4	− ·0008
$e\gamma^4$	1	0	−4	2	+ ·0002
				0	− ·0001

Haverford College :
1904 *December* 3.

Errata (additional to those given on p. 202 in Part II.)

Part I. (vol. liii.), p. 43, line 12, *for* "equal masses" *read* "masses equal to their actual masses."
" " 46, " 18, *for* "parallactic inequality" *read* "the principal parallactic inequality in longitude."
" " 83, " 2 (in some copies), *for* "$\nu_{r-\sigma,\ -i-1}$" *read* "$\nu_{r-\sigma,\ j-i-1}$."
Part II. (vol. liii.), p. 166, line 21, *for the denominator* "$u_3's_3'$" *read* "$w'_3{}^2$."
Part III. (vol. liv.), p. 5, line 6 from bottom, *for* "K^λ" *read* "K_λ."
" " 19, " 6 " *for* "+·00001 0" *read* "−·00000 9."
" " " 4 " *for* "−·00019 2" *read* "−·00012 9."
" " last line, *for* "+·19822" *read* "+·19828."

Theory of the Motion of the Moon; containing a New Calculation of the Expressions for the Coordinates of the Moon in Terms of the Time. By ERNEST W. BROWN, M.A., Sc.D., F.R.S.

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PART V. CHAPTERS X.–XV.

THE previous parts of this Memoir have been published in the *Memoirs of the Royal Astronomical Society* under the same title in 1897, 1899, 1900, 1905. They contained the solution of what I have called the main problem—the motion of the Moon under the attraction of the Earth and the Sun, which are supposed to move round one another in a fixed elliptic orbit in the plane of reference, the three bodies being treated as spheres. This fifth and last part treats of the effects of all other gravitational causes—the direct and indirect attractions of the planets, the deviations of the masses of the Earth and Moon from mechanical sphericity, and various minor perturbations which had been specifically excluded.

The problem presented by these additional perturbing forces is a complicated one. In order to clear the ground, it was first assumed that each force contained a small factor whose square could be neglected, so that the perturbations could be separately considered; then the perturbations were supposed to be expressible by a series of secular and periodic terms, each of which, under the same assumption, could also be separately considered. Chap. X. therefore contains the method for finding the effect of a perturbation expressed by a periodic or secular term, and in Chaps. XI., XII., XIII. this method is successively applied to direct and to indirect planetary action and to the action of the figures of the Earth and Moon. In Chap. XIV., in addition to minor perturbations, the effects of including the squares and higher powers of the small factor is considered: a few very small terms were found. In Chap. XV. the results from all these perturbations are gathered together, so that the final expressions for the coordinates of the Moon in terms of the time are obtained by adding the results in Chap. XV. to those previously given in Chap. IX.

The whole question of these perturbations, in spite of this division into parts, is still complex, and, in preparing the results for publication, the choice of the portions to be presented in detail was an embarrassing one. It was made more so by the conditions which rendered necessary a previous publication of the work for the direct planetary inequalities in separate form.* This work included a full investigation of the equations of variations which are needed for inequalities arising from all sources, and there was thus a choice between repeating this investigation and leaving the present Memoir incomplete. The same difficulty arose with the direct inequalities and with some other subjects which I have discussed in previous papers. I finally adopted the plan of inserting this previous work either when it was essential for clear presentation, or when it was sufficiently brief to occupy but little space, or when the proofs could be considerably improved. Thus, in Chap. X., the equations of variations, the idea of which is due to G. W. Hill, are rapidly put into the required form (*A.P.E.*, Sect. I.); the formulæ for obtaining derivatives with respect to n from a theory in which the numerical value of n'/n has been substituted (*Trans. Amer. Math. Soc.*, vol. iv.) are deduced in a few lines; the methods for dealing immediately with non-periodic terms (*Proc. Lond. Math. Soc.*, vol. xxviii., and *Trans. Amer. Math. Soc.*, vol. v.) are partly developed, but the results (*ib.*, and *Monthly Notices*, vol. lvii.) are only quoted. In Chap. XI. the proof of the theorem on which the method for the direct inequalities is based (*A.P.E.*, Sect. II.) is exhibited in a simple form, but the full algebraical results are merely quoted; for the rest of this part of the subject the methods are described in general terms and the final results alone given. In Chap. XII. a theorem for finding quickly the effects of long-period and secular inequalities in the Earth's motion is simply quoted (*Trans. Amer. Math. Soc.*, vol. vi.); this theorem was also used several times in Chap. XIV.; but the brief derivation of the disturbing function for the motion of the ecliptic (*Monthly Notices*, vol. lxviii.) is given in full.

It was found to be impracticable to present within reasonable limits much of the work actually performed. Numerous rough computations were made to find out whether coefficients or classes of coefficients were sensible; when they were insensible, a simple note, often the result of days or weeks of work, is made to that effect, but the organised plan of procedure always used in such cases is generally described; this is the case with most of the results in Chap. XIV.

Owing to the indications furnished by observation of an inequality or inequalities with a coefficient or coefficients of the order of $10''$ of arc and of very long period, one of the chief objects in view has been an investigation of such terms, and the "sieves" used in Chaps. XI., XII., XIV. were devised for this purpose. No large coefficients beyond those already known have been found. Moreover, the search has led more and more to the conclusion that no such terms can possibly arise with the laws of motion and of gravitation on which this theoretical investigation is based. If these

* *Adams Prize Essay*, Pitt Press, Cambridge, 1908. This will be referred to below by the letters *A.P.E.*

inequalities have a real existence, it would seem that the cause must be sought in some action not purely gravitational.

No part of the numerical work, except some of the multiplications of series which were necessary to find the derivatives with respect to n in Chap. X., Sect. (iii), has been turned over to computers. There are so many delicate points to consider, and so many terms and classes of terms have special peculiarities which permit the calculations to be much abbreviated, that to obtain the accuracy at which I have aimed by a general plan which could not permit these peculiarities to be used, would have involved an amount of computation out of all proportion to the final results. In fact, not more than one-third of the time occupied by these investigations has been spent on accurate numerical work. But all such work has been gone over at least twice, in many cases three times, and tested by comparisons and various methods, whenever possible.

This part concludes the theoretical investigation of the motion of the Moon under the attraction of gravitation. Its natural sequence—the formation of tables to facilitate the accurate computation of the position of the Moon at any time or for the purposes of an ephemeris—has already been arranged for, and will be undertaken at an early date. But here also it seems advisable not to set the computers at work until an extended examination of methods which will best serve the purpose, and of the properties of the final results, has been made, so that the highest possible accuracy may be obtained within the limits set by practical necessities.

The table of contents of Part V. follows.

CHAPTER X.

METHODS FOR FINDING THE REMAINING LUNAR PERTURBATIONS.

Section (i). *The Equations of Variations.*

267. *The Canonical Equations.*—The problem solved in the preceding chapters may be stated as a solution in series of the equations

$$\frac{d\mathrm{x}_i}{dt}=\frac{d\mathrm{H}}{d\mathrm{y}_i},\quad \frac{d\mathrm{y}_i}{dt}=-\frac{d\mathrm{H}}{d\mathrm{x}_i},\quad H=\tfrac{1}{2}(\mathrm{y}_1^2+\mathrm{y}_2^2+\mathrm{y}_3^2)-F \qquad (1),$$

where x_i $(i=1, 2, 3)$ are the coordinates of the Moon referred to fixed axes, and F is the force function under the hypotheses stated in Chap. I., Sect. (i).

Let w_1, w_1-w_2, w_1-w_3 be the Moon's mean longitude, and the arguments of the principal elliptic term, and of the principal term in latitude, respectively, so that w_2, w_3 are the "mean longitudes of the perigee and node" resulting from the solution of (1). Then Jacobi's method shows that, if a quantity R be added to F, and if it be expressed in terms of w_1, w_2, w_3 and c_1, c_2, c_3, the other three arbitraries of the solution of (1), the latter three may be so chosen that the solution of the differential equations

$$\frac{dc_i}{dt}=\frac{dR}{dw_i},\quad \frac{dw_i}{dt}=-\frac{dR}{dc_i}+b_i \qquad (2)$$

will give variable values of the c_i, w_i, which, when substituted in the expressions for the Moon's coordinates and velocities instead of their former values, will give the Moon's position and motion under the force function $F+R$. Here b_1, b_2, b_3 are the coefficients of t in the angles w_1, w_2, w_3. Hence, when $R=o$, $b_1=n$, the mean motion, and b_2, b_3 are the mean motions of the perigee and node. They are functions of c_1, c_2, c_3 and the constants present in the differential equations; and $b_1dc_1+b_2dc_2+b_3dc_3$ is a perfect differential.

268. *Transformation to the Variables n, c_2, c_3, w_i.*—Change to the system n, c_2, c_3, retaining the w_i unchanged, so that c_1 is now a function of n, c_2, c_3. Then

$$b_1\left(\frac{dc_1}{dn}dn+\frac{dc_1}{dc_2}dc_2+\frac{dc_1}{dc_3}dc_3\right)+b_2dc_2+b_3dc_3$$

is a perfect differential, and therefore, since $b_1=n$,

$$\frac{dc_1}{dc_2}=-\frac{db_2}{dn},\quad \frac{dc_1}{dc_3}=-\frac{db_3}{dn},\quad \frac{db_2}{dc_3}=\frac{db_3}{dc_2} \qquad (2a).$$

The equations (2) then become

$$\left.\begin{aligned}&\frac{dn}{dt}=\frac{1}{a^2\beta}\left(-\frac{dR}{dw_1}-\frac{db_2}{dn}\frac{dR}{dw_2}-\frac{db_3}{dn}\frac{dR}{dw_3}\right),\quad \frac{dc_2}{dt}=\frac{dR}{dw_2},\quad \frac{dc_3}{dt}=\frac{dR}{dw_3},\\ &\frac{dw_1}{dt}=\frac{1}{a^2\beta}\frac{dR}{dn}+b_1,\quad \frac{dw_2}{dt}=-\frac{dR}{dc_2}+b_2+\left(\frac{dw_1}{dt}-b_1\right)\frac{db_2}{dn},\quad \frac{dw_3}{dt}=-\frac{dR}{dc_3}+b_3+\left(\frac{dw_1}{dt}-b_1\right)\frac{db_3}{dn}\end{aligned}\right\} \quad (3),$$

where

$$\frac{dc_1}{dn}=-a^2\beta\,,$$

and all the functions are supposed to be expressed in terms of n, c_2, c_3, w_i. It is to be remembered throughout that, when using the variables w_i instead of the constant parts of those angles, the derivatives of w_i with respect to n, c_2, c_3 (or any functions of them) are zero.

269. *Solution of the Equations.*—It is supposed that R contains a small factor whose square may be neglected, and consequently that we may substitute the undisturbed values of n, c_2, c_3, w_i in R, that is, the values furnished by the solution of the main problem. In all the cases to be considered under this method, R can then be expressed as the sum of non-periodic and periodic terms, and each of these terms may be separately treated. First, for periodic terms, put

$$R=A'\cos(qt+q')=A'\cos(i_1w_1+i_2w_2+i_3w_3+q''t+q'''),$$

where q'', q''' are constants independent of n, c_2, c_3, w_i, and A' is a function of n, c_2, c_3 only. Substituting in the first three of equations (3) and integrating, we obtain

$$\delta n=-\frac{A'}{a^2\beta q}\frac{dq}{dn}\cos(qt+q')\,,\quad \delta c_2=\frac{i_2A'}{q}\cos(qt+q')\,,\quad \delta c_3=\frac{i_3A'}{q}\cos(qt+q') \quad . \quad . \quad (4),$$

where δn, δc_2, δc_3 are the additions to n, c_2, c_3 due to R. The arbitrary constants due to integration are given zero values.

If δb_1, δb_2, δb_3 are the corresponding changes in $b_1=n$, b_2, b_3 due to these changes in n, c_2, c_3, we have $\delta b_1=\delta n$, and from (2*a*) and (4)

$$\delta b_2=\frac{db_2}{dn}\delta n+\frac{db_2}{dc_2}\delta c_2+\frac{db_2}{dc_3}\delta c_3=\left(-\frac{1}{a^2\beta}\frac{dq}{dn}\frac{db_2}{dn}+\frac{dq}{dc_2}\right)\frac{A'}{q}\cos(qt+q')\,,$$

$$\delta b_3=\frac{db_3}{dn}\delta n+\frac{db_3}{dc_2}\delta c_2+\frac{db_3}{dc_3}\delta c_3=\left(-\frac{1}{a^2\beta}\frac{dq}{dn}\frac{db_3}{dn}+\frac{dq}{dc_3}\right)\frac{A'}{q}\cos(qt+q')\,.$$

Denoting by δw_i the additional part of w_i, substituting the value of R and these results in the second three of equations (3), and integrating, we obtain

$$\left.\begin{aligned}\delta w_1&=\frac{1}{a^2\beta}\left(\frac{1}{q}\frac{dA'}{dn}-\frac{A'}{q^2}\frac{dq}{dn}\right)\sin(qt+q')\,,\\ \delta w_2&=\left\{\left(\frac{1}{a^2\beta}\frac{dA'}{dn}\frac{db_2}{dn}-\frac{dA'}{dc_2}\right)\frac{1}{q}+\left(\frac{dq}{dc_2}-\frac{1}{a^2\beta}\frac{dq}{dn}\frac{db_2}{dn}\right)\frac{A'}{q^2}\right\}\sin(qt+q')\,,\\ \delta w_3&=\left\{\left(\frac{1}{a^2\beta}\frac{dA'}{dn}\frac{db_3}{dn}-\frac{dA'}{dc_3}\right)\frac{1}{q}+\left(\frac{dq}{dc_3}-\frac{1}{a^2\beta}\frac{dq}{dn}\frac{db_3}{dn}\right)\frac{A'}{q^2}\right\}\sin(qt+q')\end{aligned}\right\} \quad . \quad (5).$$

The equations (4), (5) constitute the theoretical solution of the problem. If λ be one of the Moon's coordinates, then the additional terms due to R are given by

$$\delta\lambda = \frac{d\lambda}{dn}\delta n + \frac{d\lambda}{dc_2}\delta c_2 + \frac{d\lambda}{dc_3}\delta c_3 + \frac{d\lambda}{dw_1}\delta w_1 + \frac{d\lambda}{dw_2}\delta w_2 + \frac{d\lambda}{dw_3}\delta w_3 .$$

270. *The Constant Term of R.*—Denoting it by R_0, we have, instead of the equations (4), δn, δc_2, δc_3 constant. These constants are at our disposal. Put $\delta n = \delta_0 n$, $\delta c_2 = 0$, $\delta c_3 = 0$. Then, instead of equations (5) we have

$$\delta w_1 = \left(\frac{1}{a^2\beta}\frac{dR_0}{dn} + \delta_0 n\right)t , \quad \delta w_2 = \left(-\frac{dR_0}{dc_2}t + \frac{db_2}{dn}\delta w_1\right), \quad \delta w_3 = \left(-\frac{dR_0}{dc_3}t + \frac{db_3}{dn}\delta w_1\right),$$

where the additive constants are made zero.

Now, since the mean longitude is a quantity observed directly, we so choose $\delta_0 n$ that w_1 is still represented by $nt + \epsilon$, and therefore $\delta w_1 = 0$. Whence the changes in the angles w_2, w_3 are obtained by adding to their motions the quantities

$$\delta b_2 = -\frac{dR_0}{dc_2}, \quad \delta b_3 = -\frac{dR_0}{dc_3} \quad \ldots \ldots \ldots \quad (6),$$

which include the change $\delta_0 n$.

Since $\delta w_1 = 0$, we have

$$\delta_0 n = -\frac{1}{a^2\beta}\frac{dR_0}{dn} \quad \ldots \ldots \ldots \quad (7).$$

This change in n must be substituted only in the *coefficients* of the periodic terms representing the Moon's coordinates, amongst them, the principal elliptic term in longitude ($2e$ with sufficient accuracy) and the principal term in latitude (2γ). These, again, are quantities observed directly, and therefore, as we wish to retain the same expressions to denote these coefficients, it is necessary to add to e, γ in all other terms the amounts

$$\delta_0 e = -\frac{de}{dn}\delta_0 n , \quad \delta_0 \gamma = -\frac{d\gamma}{dn}\delta_0 n \quad \ldots \ldots \ldots \quad (8),$$

respectively. It is true that these produce further changes in b_2, b_3, but they are quite insensible. Indeed, the changes (8) produce alterations less than $0''\cdot 01$ in any coefficient.

The changes, as found from the methods of Sect. (ii) below, give

$$\delta_0 e = -[\bar{3}\cdot 7452]\delta_0 n , \quad \delta_0\gamma = -[\bar{3}\cdot 8812]\delta_0 n .$$

If R contains a non-periodic term of the form $R_p t^p$, where R_p is independent of t, the corresponding changes in δn, δc_2, δc_3 are zero, and

$$\delta w_1 = \frac{1}{a^2\beta}\frac{dR_p}{dn}\frac{t^{p+1}}{p+1} , \quad \delta w_2 = -\frac{dR_p}{dc_2}\frac{t^{p+1}}{p+1} + \frac{db_2}{dn}\delta w_1 , \quad \delta w_3 = -\frac{dR_p}{dc_3}\frac{t^{p+1}}{p+1} + \frac{db_3}{dn}\delta w_1 .$$

Section (ii). *Reduction of the Equations to Numerical Form.*

271. *Computation of the c_i.*—The coordinates x_i of the Moon have been expressed in terms of e, k instead of c_2, c_3; we must find the relations between the two sets of constants.

Owing to the canonical forms of (1), (2), it is well known that $\Sigma_i(y_i dx_i + w_i dc_i) = dS$, a perfect differential. Hence if every quantity be expressed in terms of c_i, w_i,

$$\frac{dS}{dw_i} = \Sigma_j y_j \frac{dx_j}{dw_i}, \quad \frac{dS}{dc_i} = w_i + \Sigma_j y_j \frac{dx_j}{dc_i} \quad (i, j = 1, 2, 3).$$

But since $y_i = dx_i/dt$, these equations show that dS/dw_i consists only of cosines and $dS/dc_i - w_i$ of sines of the angles present in the solution of the main problem, and therefore that $S = \Sigma c_i w_i +$ periodic terms. Denoting by $[Q]_0$ the constant term of the expansion of Q as a sum of periodic and non-periodic terms, and substituting the value of S in the first of the previous equations, we obtain

$$c_i = \left[\Sigma_j \frac{dx_j}{dt} \frac{dx_j}{dw_i}\right]_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

the equations for finding c_i.

The co-ordinates x_1, x_2, x_3 are here those referred to fixed axes. We therefore put

$$x_1 + x_2\iota = u \text{ exp. } \iota(n't + \epsilon'), \quad x_1 - x_2\iota = s \text{ exp. } -\iota(n't + \epsilon'), \quad x_3 = z,$$

to reduce to our earlier notation.

Next, u, s have been expanded in positive and negative powers of ζ^1, ζ^c, ζ^g, ζ^m, where

$$\zeta^1 = \text{exp. } \iota(w_1 - n't - \epsilon'), \quad \zeta^c = \text{exp. } \iota(w_1 - w_2), \quad \zeta^g = \text{exp. } \iota(w_1 - w_3), \quad \zeta^m = \text{exp. } \iota(n't + \epsilon'),$$

and numerical values have not been substituted in the exponents. In line therefore with the former definition of D, I put

$$D_1 = \zeta^1 \frac{d}{d\zeta^1}, \quad D_c = \zeta^c \frac{d}{d\zeta^c}, \quad D_g = \zeta^g \frac{d}{d\zeta^g}, \quad D_m = \zeta^m \frac{d}{d\zeta^m},$$

so that

$$\frac{d}{dw_1} = \iota(D_1 + D_c + D_g), \quad \frac{d}{dw_2} = -\iota D_c, \quad \frac{d}{dw_3} = -\iota D_g,$$

and

$$D = D_1 + cD_c + gD_g + mD_m.$$

Making these substitutions in the expressions for c_i, and remembering that, as we only need the constant term, $f(u, s) + f(s, u)$ can be written $2f(u, s)$, we obtain

$$\left.\begin{array}{l} c_1 = -(n - n')[(D + 1 + m)u' \cdot (D_1 + D_c + D_g - 1)s' + Dz \cdot (D_1 + D_c + D_g)z]_0, \\ \dfrac{c_2}{n - n'} = [(D + 1 + m)u' \cdot D_c s' + Dz \cdot D_c z]_0, \quad \dfrac{c_3}{n - n'} = [(D + 1 + m)u' \cdot D_g s' + Dz \cdot D_g z]_0 \end{array}\right\} \quad (9a),$$

where $u' = u\zeta^{-1}$, $s' = s\zeta$.

As a matter of fact, c_1 will not be needed. The values of c_2, c_3 have been given in Chap. VII., § 145 (there called β_2, β_3). They are

$$\left.\begin{array}{l} \dfrac{c_2}{na^2} = -\cdot 11844\ 44e^2 - \cdot 02324e^4 - \cdot 26363e^2k^2 - \cdot 00110e^2e'^2, \\ \dfrac{c_3}{na^2} = -2\cdot 00205\ 9k^2 - 1\cdot 96376k^4 - \cdot 28546k^2e^2 - \cdot 00568k^2e'^2 \end{array}\right\} \quad . \quad . \quad (10).$$

The forms of these expressions are important: c_2 is divisible by e^2 and c_3 by k^2, and if we neglect powers of m, $c_2 = -\frac{1}{8}na^2e^2$, $c_3 = -2na^2k^2$. The numerical coefficients are

to be considered as functions of n. It must be noticed, however, that in finding the derivative of c_2 with respect to n, the terms in $\mathrm{m}^2, \mathrm{m}^3, \ldots$ diminish the value obtained from the principal term by nearly one-half; the derivative of c_3 is not much altered by these higher terms.* It is in general true that quantities depending on e, b_2 in any way are slowly convergent along powers of m, while those depending on k, b_3 are rapidly convergent.

272. *Derivatives with respect to* c_2, c_3.—These may be obtained from the derivatives with respect to e, k by solving the equations

$$\mathrm{e}\frac{dQ}{d\mathrm{e}} = \frac{dQ}{dc_2}\cdot\mathrm{e}\frac{dc_2}{d\mathrm{e}} + \frac{dQ}{dc_3}\cdot\mathrm{e}\frac{dc_3}{d\mathrm{e}}, \quad \mathrm{k}\frac{dQ}{d\mathrm{k}} = \frac{dQ}{dc_2}\cdot\mathrm{k}\frac{dc_2}{d\mathrm{k}} + \frac{dQ}{dc_3}\cdot\mathrm{k}\frac{dc_3}{d\mathrm{k}} \quad . \quad . \quad . \quad . \quad (11),$$

where Q is any function under consideration. Inserting the numerical values of e, k obtained from §§ 192, 193 of Chap. IX., I find

$$-na^2\frac{dQ}{dc_2} = +[2{\cdot}5423]\mathrm{e}\frac{dQ}{d\mathrm{e}} - [\bar{1}{\cdot}7731]\mathrm{k}\frac{dQ}{d\mathrm{k}}, \quad -na^2\frac{dQ}{dc_3} = +[2{\cdot}0929]\mathrm{k}\frac{dQ}{d\mathrm{k}} - [\bar{1}{\cdot}7386]\mathrm{e}\frac{dQ}{d\mathrm{e}} \quad . \quad (12).$$

In § 145, Chap. VII., will be found the materials to obtain b_2, b_3 (there denoted by π_1, θ_1). From them I obtain, with the help of equations (12),

$$a^2\frac{db_2}{dc_2} = +[\bar{2}{\cdot}3175], \quad a^2\frac{db_2}{dc_3} = +[\bar{2}{\cdot}3960] = a^2\frac{db_3}{dc_2}, \quad a^2\frac{db_3}{dc_3} = -[\bar{3}{\cdot}5698] \quad . \quad . \quad (13).$$

273. *Derivatives of* b_2, b_3 *with respect to* n.—These might have been obtained by finding c_1 from the first of equations ($9a$) and using the first and second of equations ($2a$). But it was found to be much more simple and sufficiently accurate to use the existing literal developments in combination with the numerical developments. The method for doing this I have given in a former paper.† It is as follows:—

Let

$$f(\mathrm{m}) = a_0 + a_1\mathrm{m} + a_2\mathrm{m}^2 + \ldots$$

and denote by $f_i(\mathrm{m})$ the sum of the first i terms of this series, and by an accent the derivative with respect to m. Then

$$\mathrm{m}f'(\mathrm{m}) = \mathrm{m}f_i'(\mathrm{m}) + i\left[a_i\mathrm{m}^i + \frac{i+1}{i}a_{i+1}\mathrm{m}^{i+1} + \frac{i+2}{i}a_{i+2}\mathrm{m}^{i+2} + \ldots\right].$$

If i be not too small, and the series not converging too slowly, the error committed by putting

$$\mathrm{m}f'(\mathrm{m}) = \mathrm{m}f_i'(\mathrm{m}) + i[f(\mathrm{m}) - f_i(\mathrm{m})]$$

will be small compared with the true value of $\mathrm{m}f'(\mathrm{m})$. When the series appears to be diminishing with fair regularity, the use of i + proper fraction instead of i in the last formula will probably give greater accuracy. As a matter of fact, I have only used this to find the greatest possible error which could have been committed, so as to avoid any sensible error in the result.

The derivatives of b_2, b_3 are found from the literal‡ and numerical values of HILL

* *A.P.E.*, p. 8. † *Monthly Notices*, vol. lvii. p. 346. ‡ *Ann. of Math.*, vol. ix. p. 40.

for the part of b_2 depending on m only, and of ADAMS * for the corresponding part of b_3; and for the other portions from the literal values of DELAUNAY † after a test by a special method which I have given earlier,‡ combined with my numerical values.

These latter portions are expressed in terms of e, γ. Since the derivatives of e/e, k/γ are insensible to the degree of accuracy required, we can make the change, after differentiating DELAUNAY'S series, by the formulæ of § 192, Chap. IX.

The required derivatives with respect to n are obtained from equations (11) combined with

$$\frac{dQ}{dn} = \left(\frac{dQ}{dn}\right) - \frac{dQ}{dc_2}\left(\frac{dc_2}{dn}\right) - \frac{dQ}{dc_3}\left(\frac{dc_3}{dn}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where the brackets denote that the enclosed functions are expressed in terms of n, e, k. The derivatives of c_2, c_3 are obtained from NEWCOMB'S transformation § of DELAUNAY'S literal values for $\mathrm{G} - \mathrm{L} = c_2$, $\mathrm{H} - \mathrm{G} = c_3$, combined with the numerical values of c_2, c_3 given above. They are only needed to two significant figures.||

The other derivatives with respect to n are found in the next section.

These apparently complicated processes are constructed to avoid the slow convergence which occurs with certain of the literal series arranged in powers of m, and I believe they have achieved the object in view.

274. The subsidiary results are

$$\left(\frac{dc_2}{dn}\right) = +[\bar{4}\cdot4637]a^2, \quad \left(\frac{dc_3}{dn}\right) = +[\bar{3}\cdot1313]a^2, \quad \left(\frac{db_2}{dn}\right) = -[\bar{2}\cdot1709], \quad \left(\frac{db_3}{dn}\right) = +[\bar{3}\cdot5736],$$

and the final results

$$\frac{db_2}{dn} = -[\bar{2}\cdot1720], \quad \frac{db_3}{dn} = +[\bar{3}\cdot5733] \quad . \quad . \quad . \quad . \quad (15).$$

Section (iii). *Derivatives with respect to n.*

275. *The equations for finding n-derivatives with the system n, c_2, c_3, w_i.*—The problem here is to find these derivatives from a theory in which the numerical value of n has been substituted.¶ I shall show that these derivatives may all be made to depend on derivatives with respect to the other five elements, and therefore with respect to e, k, w_i.

Write

$$\sum_i \left(\frac{df}{dc_i}\frac{df'}{dw_i} - \frac{df}{dw_i}\frac{df'}{dc_i}\right) = \{f, f'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Then, on changing equations (1) from the canonical set x_i, y_i to the canonical set c_i, w_i by a contact transformation, we have the relations

$$\{\mathrm{x}_i, \mathrm{y}_i\} = 1, \quad \{\mathrm{x}_i, \mathrm{y}_j\} = \{\mathrm{x}_i, \mathrm{x}_j\} = \{\mathrm{y}_i, \mathrm{y}_j\} = 0, \quad j \neq i, \quad . \quad . \quad . \quad . \quad . \quad (17).$$

* *Monthly Notices*, vol. xxxviii. p. 48.
† *C.R.*, vol. lxxiv. pp. 19 *et seq.*
‡ *Monthly Notices*, vol. lvii. p. 335.
§ *Amer. Eph. Papers*, vol. v., pt. 4 (1894), pp. 201, 202.
|| *A.P.E.*, p. 8.
¶ E. W. Brown, *Trans. Amer. Math. Soc.*, vol. iv. pp. 234–248.

Denote by brackets derivatives with respect to the system c_i, w_i. The formulæ for changing to n, c_2, c_3, w_i are

$$\frac{df}{dn}=\left(\frac{df}{dc_1}\right)\frac{dc_1}{dn}, \quad \frac{df}{dc_2}=\left(\frac{df}{dc_1}\right)\frac{dc_1}{dc_2}+\left(\frac{df}{dc_2}\right), \quad \frac{df}{dc_3}=\left(\frac{df}{dc_1}\right)\frac{dc_1}{dc_3}+\left(\frac{df}{dc_3}\right) \quad . \quad . \quad . \quad (18),$$

the other derivatives remaining unchanged.

Write (see § 268)

$$\frac{df}{dW_1}\equiv\left(\frac{df}{dw_1}-\frac{dc_1}{dc_2}\frac{df}{dw_2}-\frac{dc_1}{dc_3}\frac{df}{dw_3}\right)\div\frac{dc_1}{dn}=\left(\frac{df}{dw_1}+\frac{db_2}{dn}\frac{df}{dw_2}+\frac{db_3}{dn}\frac{df}{dw_3}\right)\div\frac{dc_1}{dn} \quad . \quad . \quad (19);$$

then $\{f, f'\}$ transforms into (f, f'), in which n replaces c_1, and W_1 replaces w_1.

It is convenient to change to our former axes by putting

$$x_1+\iota x_2=u\exp.(n't+\epsilon')\iota, \quad x_1-\iota x_2=s\exp.-(n't+\epsilon')\iota, \quad x_3=z, \quad u_1=\frac{du}{dt}, \quad s_1=\frac{ds}{dt},$$

and the transformed equations which we shall need are

$$(u_1, u)=0, \quad (u, s)=0, \quad (u, z)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Now

$$\frac{du_1}{dn}=\frac{d}{dn}\frac{du}{dt}=\frac{d}{dt}\frac{du}{dn}+\sum_i\left(\frac{du}{dw_i}\frac{db_i}{dn}\right)=\frac{d}{dt}\frac{du}{dn}+\frac{du}{dW_1}\frac{dc_1}{dn}.$$

If this be substituted in the first of equations (20), we obtain

$$\left\{U\frac{d}{dt}\frac{du}{dn}-\frac{du}{dn}\frac{dU}{dt}+U^2\right\}\div\frac{dc_1}{dn}+\frac{du}{dw_2}\frac{d}{dt}\frac{du}{dc_2}-\frac{du}{dc_2}\frac{d}{dt}\frac{du}{dw_2}+\frac{du}{dw_3}\frac{d}{dt}\frac{du}{dc_3}-\frac{du}{dc_3}\frac{d}{dt}\frac{du}{dw_3}, \quad . \quad (21)$$

where

$$U=\frac{du}{dw_1}+\frac{db_2}{dn}\frac{du}{dw_2}+\frac{db_3}{dn}\frac{du}{dw_3}.$$

Dividing by U^2 and integrating, we have the formula for du/dn:

$$\frac{du}{dn}=U\int\left(\frac{\beta a^2 Q}{U^2}-1\right)dt+CU,$$

where $-\beta a^2=\dfrac{dc_1}{dn}$, $-Q=$ last four terms of (21), $C=$ arbitrary constant.

As du/dn can contain no terms proportional to the time, the constant term of the function to be integrated must vanish, and therefore

$$-1\div\frac{dc_1}{dn}=\frac{1}{\beta a^2}=\left[\frac{Q}{U^2}\right]_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

which determines β. The arbitrary constant C is determined from the second of equations (20), and the values of db_2/dn, db_3/dn have been found in § 273. The value of dz/dn is similarly determined from the third of equations (20) when du/dn is known. The process of finding the n-derivatives is therefore reduced to an integration.

276. *Computation of the n-derivatives of u^2, us, zu, and the products of these into $(a/r)^5$.*—The process given in the preceding paragraph is neither simple in theory nor easy for computation. But, in the absence of any other method, it had to be

adopted. Various plans for abbreviating the work arose during the computations; I shall not go into these details, but shall only give the main outlines and formulæ which were used.

With the use of the notation defined in § 271, the definitions of U, Q will be slightly altered. It is convenient also to replace c_2, c_3 by $-a^2 n \mathrm{e}_2^2$, $-a^2 n \mathrm{k}_2^2$, so that e_2/e and k_2/k are positive. We put, then,

$$u' = u\zeta^{-1}, \quad c_2 = -na^2\mathrm{e}_2^2, \quad c_3 = -na^2\mathrm{k}_2^2,$$

$$U = U_0 + U_1 = (D_1 + D_e + D_g + 1)u' - \frac{db_2}{dn}D_e u' - \frac{db_3}{dn}D_g u',$$

$$Q = \tfrac{1}{2}\frac{D_e u'}{\mathrm{e}_2} \cdot \frac{d}{d\mathrm{e}_2}Du' - \tfrac{1}{2}\frac{D_e Du'}{\mathrm{e}_2} \cdot \frac{du'}{d\mathrm{e}_2} + D_g u' \cdot \frac{d}{d\mathrm{k}_2^2}Du' - D_g Du' \cdot \frac{du'}{d\mathrm{k}_2^2},$$

$$\phi = D^{-1}\left(\frac{\beta Q}{U^2} - 1 - \mathrm{m}\right), \quad n\frac{du'}{dn} = \zeta^{-1}n\frac{du}{dn} = U(\phi + C) \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

The computation of U, ϕ is made in the following way. Let U_0 consist of the terms of characteristic unity in U, so that $U_0 = (D_1 + 1)u_0'$. Then

$$\frac{1}{U^2} = \frac{1}{U_0^2} - 2\frac{U_1}{U_0^3} + 3\frac{U_1^2}{U_0^4} - \dots$$

After finding $1/U_0^i$ by special values or otherwise from the results of § 44, Chap. II., and U_1^2 by multiplication of series, and then U_1^3 for the few terms needed, multiplication of series and additions quickly give $1/U^2$; the process is simple, since $1/U_0^i$ converges rapidly, and only three or four terms are needed. Next, Q is obtained by multiplication of series quite like those used in the earlier chapters, some of which could be made use of by means of the identity $fDf' - f'Df = D(ff') - 2f'Df$. To obtain the derivatives with respect to e_2, k_2 from those for e, k, it was nearly always sufficient to multiply by e/e_2, k/k_2, respectively, as the formulæ (12) of § 272 show. Owing to the form of Q, it is necessary to use u' as far as terms with characteristics of one order higher in e, k than those needed in du/dn, but the results of previous chapters were far more than sufficient. Finally, Q/U^2 was obtained by multiplication of two series and thence β, ϕ.

The value of β as found here and tested by two other methods* is ·32962.

The second of equations (20) may be written

$$\frac{du}{dn}S - \frac{ds}{dn}U = \beta a^2\left[\frac{du}{dc_2}\frac{ds}{dw_2} - \frac{ds}{dc_2}\frac{du}{dw_2} + \frac{du}{dc_3}\frac{ds}{dw_3} - \frac{ds}{dc_3}\frac{du}{dw_3}\right].$$

Denoting by a bar the change $-\iota$ for ι, so that $S = -\bar{U}$, and using the new definition of U, we find

$$\frac{1}{\beta}U\bar{U}(2C + \phi + \bar{\phi}) = \psi = \tfrac{1}{2}\frac{D_e u'}{\mathrm{e}_2} \cdot \frac{ds'}{d\mathrm{e}_2} - \tfrac{1}{2}\frac{D_e s'}{\mathrm{e}_2} \cdot \frac{du'}{d\mathrm{e}_2} + D_g u' \cdot \frac{ds'}{d\mathrm{k}_2^2} - D_g s' \cdot \frac{du'}{d\mathrm{k}_2^2} \quad . \quad . \quad (24).$$

The multiplications of series for ψ were all at hand, and the computation of $U\bar{U}$ was simple; to determine C, only the constant terms on each side were needed.

* From the transformation of DELAUNAY'S $\mathrm{L} = c_1$ (§ 273), and from a theorem connecting it with the constant term of the parallax (E. W. BROWN, *Trans. Amer. Math. Soc.*, vol. iv. p. 247).

To find the derivative of $us = u's'$, the simplest form appeared to be

$$n\frac{d}{dn}(us) = s'\frac{du'}{dn} + u'\frac{ds'}{dn} = \tfrac{1}{2}(s'U + \overline{s'U})(\phi + \bar{\phi} + 2C) + \tfrac{1}{2}(s'U - \overline{s'U})(\phi - \bar{\phi}) \quad . \quad . \quad . \quad (25),$$

since

$$s'U - \overline{s'U} = \left\{ D_1 + \left(1 - \frac{db_2}{dn}\right)D_e + \left(1 - \frac{db_3}{dn}\right)D_g \right\} us,$$

and the product us was at hand, the constant disappearing, so that all the terms were small. For the purposes of verification

$$(s'U + \overline{s'U})(\phi + \bar{\phi} + 2C) = \left\{ \frac{u'}{U} + \left(\frac{u'}{U}\right) \right\} \psi\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

was computed: this gives the more important part of $d(us)/dn$.

The derivative of u'^2 is given by

$$n\frac{d}{dn}u'^2 = 2u'U(\phi + C) = (\phi + C)\left(D_1 + D_e + D_g + 1 - \frac{db_2}{dn}D_e - \frac{db_3}{dn}D_g\right)u'^2 \quad . \quad . \quad (27),$$

for which the product u'^2 was available. A multiplication of series gives the required function.

The formula for dz/dn is obtained from the third of equations (20). It is transformed into

$$\left.\begin{aligned} n\frac{dz}{dn} &= Z(\phi + C) - \frac{\beta}{U}\chi, \quad Z = \left(D_1 + D_e + D_g - \frac{db_2}{dn}D_e - \frac{db_3}{dn}D_g\right)z, \\ \chi &= \tfrac{1}{2}\frac{D_e z}{e_2}\cdot\frac{du'}{de_2} - \tfrac{1}{2}\frac{D_e u'}{e_2}\cdot\frac{dz}{de_2} + D_g z\cdot\frac{du'}{dk^2} - \tfrac{1}{2}\frac{D_g u'}{k}\cdot\frac{dz}{dk} \end{aligned}\right\} \quad . \quad . \quad (28),$$

the multiplications of series for χ being at hand.

The derivative of $u'z$ is therefore given by

$$n\frac{d}{dn}(u'z) = -\frac{\beta u'}{U}\chi + (\phi + C)\left(D_1 + D_e + D_g + 1 - \frac{db_2}{dn}D_e - \frac{db_3}{dn}D_g\right)u'z \quad . \quad . \quad . \quad (29),$$

which is computed as before. Only the principal terms were required.

The derivative of z^2 is given by (28), if we replace z by z^2 in the formulæ.

The few terms for derivatives of third-degree products of u, s, z were computed from these results. The numerical values are given in Sect. (v).

277. *To find the derivatives of the products of u^2, us, uz, with $(a/r)^5$*, we only further need the derivative of the last. For the only term which is sensible we can use DELAUNAY's results, combined with the numerical results of § 266, Chap. IX. This is because the constant term of a/r contains no portion depending on e^2, γ^2, and the portions depending on m converge very rapidly; the other terms in the products are of orders e^2, γ^2, m^4, e^2m^2, $m^2\gamma^2$, or of higher orders. The results from all but those of orders e^2, γ^2, are practically insensible.

Section (iv). *The Final Form of the Equations of Variations.*

278. *Numerical values required.*—I gather together the numerical results obtained in the preceding sections, so far as they are needed in the equations of variations. They are as follows: all numbers whose common logarithms are set down being enclosed in square brackets:—

$$\left.\begin{array}{l} e=+[\bar{2}\cdot7396],\quad \mathrm{e}=+[\bar{1}\cdot0396],\quad \gamma=+[\bar{2}\cdot6521],\quad \mathrm{k}=+[\bar{2}\cdot6511],\\ \beta=+[\bar{1}\cdot51801],\quad m=\dfrac{n'}{n}=+[\bar{2}\cdot87391],\quad \mathrm{m}=\dfrac{n'}{n-n'}=+[\bar{2}\cdot90768],\quad \dfrac{\mathrm{a}^2}{a^2}=+[\bar{1}\cdot99921] \end{array}\right\} \qquad (30);$$

$$\left.\begin{array}{l} \dfrac{db_2}{dn}=-[\bar{2}\cdot1720],\quad a^2\dfrac{db_2}{dc_2}=+[\bar{2}\cdot3175],\quad a^2\dfrac{db_2}{dc_3}=+[\bar{2}\cdot3960],\\ \dfrac{db_3}{dn}=+[\bar{3}\cdot5733],\quad a^2\dfrac{db_3}{dc_2}=+[\bar{2}\cdot3960],\quad a^2\dfrac{db_3}{dc_3}=-[\bar{3}\cdot5698] \end{array}\right\} \quad . \quad . \quad . \quad (31),$$

$$-na^2\frac{dA'}{dc_2}=+[2\cdot5423]\mathrm{e}\frac{dA'}{d\mathrm{e}}-[\bar{1}\cdot7731]\mathrm{k}\frac{dA'}{d\mathrm{k}},\quad -na^2\frac{dA'}{dc_3}=+[2\cdot0929]\mathrm{k}\frac{dA'}{d\mathrm{k}}-[\bar{1}\cdot7386]\mathrm{e}\frac{dA'}{d\mathrm{e}} \qquad (32).$$

279. *The Equations of Variations.*—The equations are now put into a form which is to admit of direct application to any term of the disturbing function. The coefficient of each cosine or sine which constitutes the variation of an element is to be expressed in seconds of arc. Put

$$R=\frac{1}{4}\frac{m''}{m'}n'^2\mathrm{a}^2A\cos(i_1w_1+i_2w_2+i_3w_3+q''t+q')=\frac{1}{4}\frac{m''}{m'}n'^2\mathrm{a}^2A\cos(qt+q') \quad . \quad . \quad (33);$$

s' = number of seconds of arc in the daily mean motion of the Sun $=3548''\cdot19$,

s = ,, ,, ,, ,, argument $qt+q'$;

$$f=\frac{1}{4}\frac{m''}{m'}\frac{\mathrm{a}^2}{a^2}\frac{\mathrm{s}'^2}{\beta}206265=+[12\cdot29358]\frac{m''}{m'},\quad f'=\frac{1}{4}\frac{m''}{m'}\frac{\mathrm{a}^2}{a^2}\frac{m\mathrm{s}'}{\beta}=+[7\cdot61748]\frac{m''}{m'} \quad . \quad (34);$$

$$C=\frac{fA}{\mathrm{s}^2},\quad C_1=\frac{f'A}{\mathrm{s}},\quad j_1A=\frac{n}{\mathrm{a}^2}\frac{d}{dn}(A\mathrm{a}^2),\quad j_2A=\mathrm{e}\frac{dA}{d\mathrm{e}},\quad j_3A=\mathrm{k}\frac{dA}{d\mathrm{k}} \quad . \quad . \quad . \quad . \quad (35);$$

$$\left.\begin{array}{l} \lambda_1=-i_1+\cdot01486i_2-\cdot003744i_3,\\ \lambda_2=+\cdot01486i_1+\cdot006624i_2+\cdot008260i_3,\\ \lambda_3=-\cdot003744i_1+\cdot008260i_2-\cdot001238i_3 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$\left.\begin{array}{l} \mu_2=-[\bar{2}\cdot1720]j_1+[2\cdot0603]j_2-[\bar{1}\cdot2911]j_3,\\ \mu_3=+[\bar{3}\cdot5733]j_1-[\bar{1}\cdot2566]j_2+[1\cdot6109]j_3 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37);$$

then the equations of variations are

$$\left.\begin{array}{ll} \dfrac{\delta n}{n}=\lambda_1C_1\cos(qt+q'), & \delta w_1=(\lambda_1C+j_1C_1)\sin(qt+q'),\\ \dfrac{\delta c_2}{na^2}=+[\bar{1}\cdot51801]i_2C_1\cos(qt+q'), & \delta w_2=(\lambda_2C+\mu_2C_1)\sin(qt+q'),\\ \dfrac{\delta c_3}{na^2}=+[\bar{1}\cdot51801]i_3C_1\cos(qt+q'), & \delta w_3=(\lambda_3C+\mu_3C_1)\sin(qt+q') \end{array}\right\} \quad . \quad . \quad (38).$$

If it be desired to find δe, $\delta\gamma$, they can be obtained from

$$\delta e=-[1\cdot2823]\frac{\delta c_2}{na^2}+[\bar{3}\cdot43]\frac{\delta c_3}{na^2}+[\bar{3}\cdot7452]\frac{\delta n}{n},\quad \delta\gamma=-[\cdot7460]\frac{\delta c_3}{na^2}-[\bar{1}\cdot2232]\frac{\delta c_2}{na^2}+[\bar{3}\cdot8812]\frac{\delta n}{n} \qquad (39).$$

These formulæ are not easily comparable with those of HILL,* RADAU,† or NEWCOMB,‡ because I use derivatives with respect to n on the assumption that A' is expressed in terms of n, c_2, c_3, while they suppose that the coefficient is expressed in terms of n, c, γ. The chief gain here is the avoidance of the doubtful derivative dc_2/dn, which in my method is certain to the degree of accuracy required; and further, the method in Sect. (iii) of finding the n-derivatives is more simple with the system n, c_2, c_3 than with the other system. The derivative on which nearly all coefficients depend is $\frac{dc_1}{dn} = -\beta a^2$, and this is found accurately. The comparison, however, can be made for all terms except those in δw_i involving C_1: this has been partly done.§

280. *Method of using the equations and abbreviations.*—The numbers f, f' are the same for all the direct perturbations of a given planet; for all indirect perturbations; they have two values for terms dependent on the motion of the ecliptic; and three values for terms depending on the figure of the Earth. The numbers λ_1, λ_2, λ_3, μ_2, μ_3 are the same for a particular Moon argument in a given class of perturbations; the first and third terms of μ_2 and the first and second terms of μ_3 can nearly always be neglected. Only two numbers, C, C_1, have to be computed for each coefficient required, and one of these is generally very small.

The most important particular cases are:—

(*a*) $i_1 = i_2 = i_3 = 0$. Then $\delta n = \delta c_2 = \delta c_3 = 0$ and $\lambda_i C = 0$ in the δw_i.

(*b*) *Terms of long period in which* $i_1 \neq 0$. The portions depending on C_1 are very small.

(*c*) *Terms of short period approximating to a month or less.* The portions depending on C are usually small compared with those depending on C_1, and it is rarely necessary to compute δc_2, δc_3, owing to the theorem in the next section; δn is very small.

(*d*) *Terms depending on* w_2 *or* w_3, *but not on* w_1. These are much the most troublesome to compute, because δc_2, δw_1, δw_2, or δc_3, δw_1, δw_3, may produce terms of the same order in the Moon's coordinates. But even here δn is usually insensible, and the theorem given in the next paragraph is almost exactly satisfied, so that the computations reduce to finding δw_1, δw_2 (and δc_2 as a test only), or δw_1, δw_3 (and δc_3 as a test).

281. *Substitution of the elements in the coordinates.*—We have, for the longitude,

$$\delta V = \delta w_1 + \left(\frac{dV}{dw_1} - 1\right)\delta w_1 + \frac{dV}{dw_2}\delta w_2 + \frac{dV}{dw_3}\delta w_3 + \frac{dV}{dn}\delta n + \frac{dV}{dc_2}\delta c_2 + \frac{dV}{dc_3}\delta c_3 \qquad . \qquad . \qquad (40).$$

The terms arising from the first in this expression, δw_1, are called *primary*; those from the others, that is, from the periodic terms, *secondary.*‖ All terms in latitude are secondary, and there are scarcely any sensible terms in parallax.

* *Wash. Astr. Papers*, vol. iii., and *Coll. Works*, vol. ii. p. 336.

† *Ann. Obs. Paris*, vol. xxi.

‡ *Carnegie Inst. Publ.*, No. 72.

§ *A.P.E.*, p. 10; *Monthly Notices*, vol. lxviii. p. 167.

‖ *A.P.E.*, p. 37.

Since δn is rarely sensible, it is sufficient to obtain dV/dn from DELAUNAY'S* final results. The values of dV/dc_2, dV/dc_3 are rarely required, except for the purposes of a test, owing to the following

282. *Theorem.*†—*If* δw_2 *be confined to the term* $\mu_2 C_1 \sin(qt+q')$ *and a coordinate* λ *to the term* $p \sin(i_2' w_2 + \psi')$, *where* ψ' *is independent of* w_2 *and* i_2' *has the same sign as* i_2, *and* $i_2' \neq 0$, $i_2 \neq 0$, *then the variation of the coordinate* λ *due to* δw_2, δc_2 *is*

$$\frac{d\lambda}{dc_2}\delta c_2 + \frac{d\lambda}{dw_2}\delta w_2 = \mu_2 C_1 p i_2' \sin\{qt + q' - i_2' w_2 - \psi'\}.$$

That this theorem is true when we neglect all but the lowest power of e present in the coefficients and when δn is negligible is immediately seen. For, in such cases, $j_2 = |i_2|$, δc_2 depends on dR/dw_2, and δw_2 on dR/dc_2. But it appears to hold even more accurately. An exactly similar theorem holds for δc_3, δw_3, and the terms depending on w_3 in a coordinate. In the case of the principal term in latitude due to the figure of the Earth it holds within one-tenth of one per cent. of the whole.

Section (v). *Numerical Values of Functions of the Lunar Coordinates.*

283. These functions are chiefly products of the second order in u, s, z. The computation of them by multiplications of series is quite like those necessary in previous chapters, and indeed the great majority of them had been obtained in the solution of the main problem. The formation of the derivatives with respect to n, e, k has been developed in previous sections. It is understood that the n derivatives are formed with

$$j_1 C\,.\,\text{char.} = \frac{n}{\mathrm{a}^2}\frac{d}{dn}(C\mathrm{a}^2\,.\,\text{char.}),$$

where C is any one of these coefficients (given a numerical value here) expressed as a function of n, c_2, c_3, except in the case of the figure of the Earth terms which are formed with $j_1 nC\,.\,\text{char.} = d(Cn^2\,.\,\text{char.})/dn$, and the case of terms due to the motion of the ecliptic which are formed with $j_1 \mathrm{a}^2 C\,.\,\text{char.} = d(n\mathrm{a}^2 C\,.\,\text{char.})/dn$. The few coefficients where an exact computation of these latter forms was necessary are given; in general, these derivatives were not needed, or their values could be set down from the known principal term in the literal expansion of the coefficient.

The values of $(a/r)^3$, $(a/r)^5$ needed for the figure of the Earth terms are obtained directly from the numerical values of a/r in § 266, Chap. IX. These are simple to compute for the terms needed, owing to the rapid convergence of the parallax terms. The formation of the derivatives has been explained in § 277.

284. The values of V, U needed to find δV, δU are obtained directly from the results of Chap. IX., after division of the coefficients there given by 206265. The fact that δn is always small, combined with the theorem of § 282, and the expressions in § 279 of δe, $\delta\gamma$ in terms of δc_2, δc_3, make the computation of the derivatives of V

* *Mém. de l'Acad. d. Sc.*, vol. xxix. chap. 9. † *A.P.E.*, p. 15.

with respect to e, γ unnecessary; they can, however, be found from the results of Sect. (v), Chap. IX., if needed. The value of dV/dn so far as it is wanted is given by

$$n\frac{dV}{dn} = +[\bar{2}\cdot0465] \sin l - [\bar{2}\cdot4669] \sin (2D - l) - [\bar{2}\cdot3965] \sin 2D \, . \qquad (41).$$

285. *The coefficients M_i.*—These are defined by the following equations, in which θ is one of the angles and M_i the corresponding coefficient in the expansions of the functions set down. Summations for all such terms constitute the complete values of the functions.

According to the previous notation, V, V' are the ecliptic true longitudes of the Moon and Sun, $r^2 = us + z^2 = \rho^2 + z^2$, T = the mean longitude of the Sun, etc. Then

$$\frac{a'^2}{r'^2}\,\frac{r^2 - 3z^2}{\mathrm{a}^2} = 2M_1 \cos\theta, \quad \frac{a'^2}{r'^2}\,\frac{\rho^2}{\mathrm{a}^2}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\, 2(V - V') = \begin{matrix}M_2 \cos\\ M_3 \sin\end{matrix}\,\theta, \quad \frac{a'^2}{r'^2}\,\frac{z\rho}{\mathrm{a}^2}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\,(V - h'') = \begin{matrix}M_4 \sin\\ -M_4 \cos\end{matrix}\,\theta,$$

$$\frac{a'^3}{r'^3}\,\frac{\rho(r^2 - 5z^2)}{\mathrm{a}^2}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\,(V - V') = \begin{matrix}M_6 \cos\\ M_7 \sin\end{matrix}\,\theta, \quad \frac{1}{3}\,\frac{a'^3}{r'^3}\,\frac{\rho^3}{\mathrm{a}^3}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\, 3(V - V') = \begin{matrix}M_8 \cos\\ M_9 \sin\end{matrix}\,\theta,$$

$$\frac{a'^3}{r'^3}\,\frac{z(r^2 - \frac{5}{3}z^2)}{\mathrm{a}^3}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\,(V - h'') = \begin{matrix}M_{10} \sin\\ -M_{10} \cos\end{matrix}\,\theta, \quad \frac{a'^3}{r'^3}\,\frac{z\rho^2}{\mathrm{a}^3}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\,(2V - V' - h'') = \begin{matrix}M_{12} \sin\\ -M_{12} \cos\end{matrix}\,\theta.$$

For computation purposes these transform into

$$\frac{a'^2}{r'^2}\,\frac{us - 2z^2}{\mathrm{a}^2} = M_1(e^{\theta\iota} + e^{-\theta\iota}), \quad \frac{a'^2}{r'^2}\,\frac{u^2}{\mathrm{a}^2}e^{2(\mathrm{T} - V')\iota} = \tfrac{1}{2}(M_2 \pm M_3)e^{\pm\theta\iota}, \quad \frac{a'^2}{r'^2}\,\frac{z\iota\cdot u}{\mathrm{a}^2}e^{(\mathrm{T} - h'')\iota} = M_4 e^{\theta\iota},$$

$$\frac{a'^3}{r'^3}\,\frac{u(us - 4z^2)}{\mathrm{a}^2}e^{(\mathrm{T} - V')\iota} = \tfrac{1}{2}(M_6 \pm M_7)e^{\pm\theta\iota}, \quad \frac{1}{3}\,\frac{a'^3}{r'^3}\,\frac{u^3}{\mathrm{a}^3}e^{3(\mathrm{T} - V')\iota} = \tfrac{1}{2}(M_8 \pm M_9)e^{\pm\theta\iota},$$

$$\frac{a'^3}{r'^3}\,\frac{\iota z\cdot(us - \frac{2}{3}z^2)}{\mathrm{a}^3}e^{(V' - h'')\iota} = M_{10}e^{\theta\iota}, \quad \frac{a'^3}{r'^3}\,\frac{\iota z\cdot u^2}{\mathrm{a}^3}e^{(2\mathrm{T} - V' - h'')\iota} = M_{12}e^{\theta\iota}.$$

No values for M_{10}, M_{12} are given below, as there were no terms large enough to make their accurate computation necessary. The computations of these coefficients from the results given are quite simple. Besides the operations already mentioned, there are multiplications by a'^2/r'^2, $(a'^2/r'^2) \cos 2V'$, etc., but the simplicity and brevity of these processes make further detail unnecessary.

286. The coefficients marked with a dagger (†) in the following tables include characteristics of two orders higher than the principal characteristic, and are therefore fully accurate to four significant figures. All other coefficients were sufficiently accurate with the part depending only on the principal characteristic.

287. *Coefficient of* $\zeta^{\theta+2i}\times$ *characteristic* = *coef. of* $\zeta^{-\theta-2i}\times$ *char., in*

Char.	θ	$\frac{us-2z^2}{a^2}$ $i=0$	$i=1$	$i=-1$	$\frac{n}{a^2}\frac{d}{dn}(us-2z^2)$ $i=0$	$i=1$	$i=-1$
1	0	+ ·99262†	− ·00701†	− ·00701†	− 1·32383†	+ ·02538†	+ ·02538†
e'	m	+ ·00787†	+ ·00410†	− ·02992†	− ·0240 †	− ·0151 †	+ ·1140 †
e'^2	2m	+ ·0101 †	+ ·0005 †	− ·0878 †	− ·022	·000	+ ·354
e	c	− ·49073†	− ·00262†	− ·08581†	+ ·5925	− ·0008	+ ·2220
ee'	c + m	+ ·1455 †	+ ·0016 †	− ·2090 †	− ·4112	− ·0070	+ ·5163
ee'	c − m	− ·1829 †	− ·0088 †	+ ·0338 †	+ ·5619	+ ·0348	+ ·0531
ee'^2	c + 2m	+ ·0984		− ·4034	− ·264	+ ·001	+ ·991
ee'^2	c − 2m	− ·1817	− ·0282	+ ·2009	+ ·601	+ ·116	− ·756
ee'^3	c + 3m	+ ·01		− ·85			
ee'^3	c − 3m	+ ·06	+ ·28	+ ·04			
e^2	2c	− ·0618	− ·0007	+ ·0758	+ ·0986	+ ·0008	− ·2228
e^2e'	2c + m	+ ·0371	+ ·0008	+ ·1794	− ·100	− ·003	− ·440
e^2e'	2c − m	− ·0460	− ·0035	+ ·0027	+ ·131	+ ·012	− ·177
k^2	2g	+ 2·9897	+ ·0090	− ·2062	− 2·9533	− ·0310	+ ·4638
k^2e'	2g + m	− ·1092	− ·0057	− ·5989	+ ·018	+ ·018	+ 1·503
k^2e'	2g − m	+ ·0985	+ ·0409	+ ·2296	+ ·016	− ·145	− ·526

Char.	θ	$-\frac{2z^2}{a^2}$ $i=0$	$i=1$	$i=-1$	$\frac{n}{a^2}\frac{d}{dn}(-2z^2)$ $i=0$	$i=1$	$i=-1$
1	0	− ·00803	+ ·00028	+ ·00028	+ ·00800	− ·00064	− ·00064
e'	m	− ·00005	− ·00033	+ ·00083	+ ·0001	+ ·0007	− ·0021
e	c	+ ·00386	+ ·00014	+ ·00029	− ·0034	− ·0030	− ·0055
ee'	c + m	− ·0014	− ·0002	+ ·0007			
ee'	c − m	+ ·0012	+ ·0004	·0000			
k^2	2g	+ 1·9997	+ ·0060	− ·1479	− 1·9888	− ·0208	+ ·3396
k^2e'	2g + m	− ·0646	− ·0038	− ·4377	− ·012	+ ·012	+ 1·116
k^2e'	2g − m	+ ·0762	+ ·0273	+ ·1608	− ·020	− ·096	− ·376

Char.	θ	$e\frac{d}{de}\left(\frac{us-2z^2}{a^2}\right)$ $i=0$	$i=1$	$i=-1$
1	0	+ ·00922	− ·00057	− ·00057
e'	m	+ ·00183	+ ·00043	− ·00177
e'^2	2m	+ ·0021	+ ·0013	− ·0042

For other arguments,

$$e\frac{d}{de}\left(\frac{us-2z^2}{a^2}\right)=\frac{us-2z^2}{a^2}\times\text{index of e in char.},$$

with sufficient approximation.

$$k\frac{d}{dk}\left(\frac{us-2z^2}{a^2}\right)=-\frac{6z^2}{a^2}$$

with sufficient approximation for terms not containing argument 2g.

$$k\frac{d}{dk}\left(\frac{us-2z^2}{a^2}\right)=\frac{us-2z^2}{a^2}\times\left(\begin{array}{c}\text{index of}\\ \text{k in char.}\end{array}\right)$$

for terms containing argument 2g.

288. *Value of* $\frac{1}{a^2}u^2\zeta^{-2}$. *Coefficient of* $\zeta^{\theta+2i}\times$ *characteristic.*

Char.	θ	$i=2$	$i=1$	$i=0$	$i=-1$	$i=-2$
1	0		+ ·0043†	+ ·9879†	− ·0140†	+ ·0001†
e'	m		− ·0034†	− ·1860†	− ·0612†	+ ·0006†
e'	− m	+ ·0002†	+ ·0182†	+ ·1961†	+ ·0056†	− ·0001†
e'^2	2m		− ·0026†	− ·1039†	− ·1817†	+ ·0031†
e'^2	− 2m	+ ·0013†	+ ·0553†	+ ·1535†	− ·0066†	
e	c	+ ·0001†	+ ·0046†	+ ·4979†	− ·2982†	+ ·0024†
e	− c	+ ·0010†	+ ·1072†	− 1·4933†	+ ·0123†	
ee′	c + m		− ·0039	− ·2342	− ·7475	+ ·0167
ee′	− c − m	+ ·0057	+ ·3234	+ ·1403	− ·0090	
ee′	c − m	+ ·0003	+ ·0186	+ ·3030	+ ·0499	− ·0019
ee′	− c + m	− ·0009	− ·0581	− ·3106	+ ·0576	
ee′²	c + 2m			− ·11	− 1·53	+ ·07
ee′²	− c − 2m	+ ·03	+ ·75	+ ·13		
ee′²	c − 2m		+ ·06	+ ·34	+ ·57	
ee′²	− c + 2m		− ·20	− ·35	+ ·18	
e²	2c		+ ·0033	+ ·2507	− ·2069	+ ·0259
e²	− 2c	+ ·0112	− ·0519	+ ·6234	− ·0002	
e²e′	2c + m			− ·19	− ·45	+ ·13
e²e′	− 2c − m	+ ·06	− ·15	− ·24		
e²e′	2c − m		+ ·02	+ ·25	− ·11	− ·01
e²e′	− 2c + m	− ·01	+ ·07	+ ·37		
k²	2g			+ ·0030	− ·1861	+ ·0033
k²	− 2g	+ ·0005	+ ·0896	+ 1·9743	− ·0142	
k²e′	2g + m				− ·41	+ ·02
k²e′	− 2g − m		+ ·23	+ ·28	+ ·01	
k²e′	2g − m		− ·16	− ·34	− ·06	
k²e′	− 2g + m				+ ·24	− ·01

Value of $\frac{k}{a^2}\frac{d}{dk}(u^2\zeta^{-2})$

Char.	θ	$i=1$	$i=0$	$i=-1$
1	0	− ·00005	− ·00806	+ ·00073
e'	m	+ ·0001	+ ·0046	+ ·0022
e'	− m	− ·0002	− ·0050	− ·0006
e'^2	2m		+ ·002	+ ·005
e'^2	− 2m	− ·001	− ·004	

For other arguments, multiply by power of k in characteristic.

Value of $\frac{n}{a^2}\frac{d}{dn}(u^2\zeta^{-2})$. *Coefficients of* $\zeta^{\theta+2i}\times$ *characteristic.*

Char.	θ	$i=2$	$i=1$	$i=0$	$i=-1$	$i=-2$
1	0	− ·00014†	− ·01454†	− 1·31548†	+ ·05410†	− ·00040†
e'	m	+ ·0002 †	+ ·0104 †	+ ·3552 †	+ ·2420 †	− ·0040 †
e'	− m	− ·0014 †	− ·0686 †	− ·3784 †	− ·0254 †	+ ·0006 †
e'^2	2m			+ ·116	+ ·820	− ·022
e'^2	− 2m	− ·002	− ·1[illegible]4	− ·234		
e	c	− ·0002	− ·0133	− ·6255	+ ·7981	− ·0122
e	− c	− ·0039	− ·2956	+ 1·8414	− ·0450	+ ·0001
ee'	c + m	+ ·0004	+ ·0164	+ ·5468	+ 2·0102	− ·0842
ee'	− c − m	− ·0304	− ·9260	− ·6502	+ ·0366	− ·0002
ee'	c − m	− ·0014	− ·0738	− ·8300	+ ·3428	+ ·0058
ee'	− c + m	+ ·0030	+ ·0148	+ 1·3722	− ·2232	
ee'^2	c + 2m			+ ·190	+ 4·344	− ·364
ee'^2	− c − 2m	− ·140	− 2·292	− ·616	+ ·004	
ee'^2	c − 2m		− ·272	− 1·064	− 2·092	+ ·030
ee'^2	− c + 2m	+ ·008	+ ·784	+ 1·650	− ·748	
e^2	2c	− ·0002	− ·0116	− ·2846	+ ·5452	− ·1028
e^2	− 2c	− ·0458	+ ·1216	− ·7044	+ ·0006	
e^2e'	2c + m		+ ·016	+ ·458	+ 1·130	− ·540
e^2e'	− 2c − m	− ·268	+ ·348	+ ·740		
e^2e'	2c − m	− ·002	− ·070	− ·676	+ ·976	− ·022
e^2e'	− 2c + m	+ ·008	− ·270	− 1·306	+ ·006	
k^2	2g		− ·0002	− ·0076	+ ·3332	− ·0112
k^2	− 2g	− ·0022	− ·1502	− 1·9150	+ ·0456	− ·0002
k^2e'	2g + m			+ ·004	+ ·726	− ·070
k^2e'	− 2g − m	− ·014	− ·374	− ·484	− ·020	+ ·002
k^2e'	2g − m			− ·008	− ·504	+ ·022
k^2e'	− 2g + m	+ ·004	+ ·366	+ ·680	+ ·194	+ ·002

Value of $\frac{e}{a^2}\frac{d}{de}(u^2\zeta^{-2})$

1	0	+ ·00003	+ ·00249	− ·01584	+ ·00612
e'	m		− ·0021	− ·0090	+ ·0176
e'	− m	+ ·00003	+ ·0082	+ ·0025	− ·0028
e'^2	2m		− ·005	− ·009	+ ·040
e'^2	− 2m	+ ·001	+ ·020	+ ·004	− ·012

For other arguments, multiply by power of e in characteristic.

289. *Coefficient of ζ^θ × characteristic in*

$\frac{u\zeta^{-1}\cdot z\iota}{a^2}$ and $\frac{n}{a^2}\frac{d}{dn}(u\zeta^{-1}\cdot z\iota)$

Char.	θ	Coef.	
k	g	+ ·998	− 1·1646
k	− g	− 1·0003	+ 1·1681† (for the three rows k, k^3, ke^2 with − g)
k^3	− g	+ 1·99	
ke^2	− g	− ·503	
k	g + 2	+ ·0030	− ·0108
k	− g − 2	+ ·0072	− ·0246
k^3	− g − 2	− ·035	
ke^2	− g − 2	+ ·063	
k	g − 2	− ·0457	+ ·1213
k	− g + 2	+ ·0355	− ·0857
ke^2	− g + 2	+ ·027	
ke	g + c	+ ·503	
ke	− g − c	+ ·496	
ke	g − c	− 1·487	
ke	− g + c	+ ·480	
ke	g + c − 2	− ·280	
ke	− g − c + 2	+ ·036	
ke	g − c + 2	+ ·108	
ke	− g + c − 2	+ ·088	
ke	g − c − 2	+ ·026	
ke	− g + c + 2	+ ·015	
ke′	g + m	− ·1082	+ ·1439
ke′	g + m − 2	− ·1430	+ ·6456
ke′	g − m	+ ·1176	− ·1755
ke′	g − m − 2	+ ·0416	− ·1805
ke′	− g − m	− ·0835	+ ·1745
ke′	− g − m + 2	+ ·1066	− ·2825
ke′	− g + m	+ ·0718	− ·1384
ke′	− g + m + 2	− ·0433	+ ·1084
ke′	− g + m − 2	+ ·0299	− ·1077
ke^2	− g − 2c − 2	− ·0766	
ke^2	g + 2c − 4	+ ·0198	

290. *Coef. of ζ^θ × char. in*

$\left(\frac{a}{r}\right)^5$

Char.	±θ	Coef.
1	0	+ 1·0046
e^2	0	+ 1·286
k^2	0	·00
1	2	+ ·01794
e^2	2	+ ·481
k^2	2	·00
e	c	+ 1·262
e	c + 2	+ ·0427
e	c − 2	+ ·2504
e	c − 4	+ ·0081
e^2	2c	+ 1·244
e^2	2c − 2	+ ·2208
e^2	2c − 4	+ ·048
k^2	2g	·0000
k^2	2g − 2	− ·039

Coef. of ζ^θ in

$$\frac{a^5(r^2-3z^2)}{r^5 a^2}$$

0	+ ·994
2	+ ·0132
c	+ ·0823
c − 2	+ ·0158
2c − 2	+ ·00072
2g	+ ·00613
2g − 2	− ·00036

Coef. of ζ^θ × char. in

$$\frac{u^2}{a^2}\left(\frac{a}{r}\right)^5$$

Char.	θ	
1	2	+ ·9915
e	c	− ·0697
e	− c + 2	− ·268
e	c + 2	+ 1·723
e^2	− 2c + 2	+ ·004
k^2	− 2g + 2	+ 2·013

$n^2 C\mathrm{k}$ = *Coef. of* ζ^{-g} *in*

$$\frac{n^2\cdot u\zeta^{-1}\cdot z\iota}{a^2}\left(\frac{a}{r}\right)^5$$

C = − 1·0062 †
j_1 = − 2·1566 †
j_2 = + ·01217†
j_3 = + ·9920 †

291. *Coefficients of ζ^θ in*

$$\frac{2u\zeta^{-1}\cdot D(\iota z)}{\mathrm{k}a^2(1+\mathrm{m})} - \frac{(D+1+\mathrm{m})(u\zeta^{-1}\cdot\iota z)}{\mathrm{k}a^2(1+\mathrm{m})}$$

θ	Coef.
g	+ ·003
− g	+ 2·0052†
g + 2	− ·001
− g − 2	+ ·006
g − 2	+ ·0529
− g + 2	+ ·0017

With factor na^2k and argument $\theta = -$g

j_1 − ·170†
j_2 + ·00026
j_3 + 1·0000†

292. *Values of* M_i, j_1M_i, j_2M_i, j_3M_i.

To obtain the values, each coefficient is to be multiplied by its characteristic.

M_1, M_2 are unaltered, and M_3 changes sign when the angle changes sign.

Coefficients followed by the mark † have characteristics of two orders higher included.

Char.	Angle.	M_1	M_2	M_3	j_1M_1	j_1M_2	j_1M_3
1	0	+ ·99276†	− ·0280†	0	− 1·3240†	+ ·1080†	0
e'	l'	+ 1·0005†	− ·0835†	− ·0107†	− 1·3478†	+ ·3245†	+ ·0509†
e'^2	$2l'$	+ 1·259†	− ·201†	− ·040†	− 1·701	+ ·852	+ ·302
e'^3	$3l'$	+ 1·65			− 2·21	+ 2·0	0·0
e	l	− ·49080†	− ·2862†	− ·3108†	+ ·5926	+ ·7523	+ ·8423
ee'	$l+l'$	− ·3542†	− ·421	− ·477	+ ·1813	+ 1·1117	+ 1·3085
ee'	$l-l'$	− ·6736†	− ·799	− ·889	+ 1·1544	+ 2·5564	+ 2·9124
ee'^2	$l+2l'$	− ·370	− ·73	− ·83	+ ·065	+ 2·151	+ 2·509
ee'^2	$l-2l'$	− ·978	− 1·10	− 1·34	+ 1·904	+ 3·595	+ 4·643
e^2	$2l$	− ·0618	− ·2069	− ·2065	+ ·0986	+ ·5453	+ ·5441
e^2e'	$2l+l'$	− ·0247	− ·24	− ·24	− ·001	+ ·586	+ ·582
e^2e'	$2l-l'$	− ·1078	− ·73	− ·73	+ ·230	+ 2·614	+ 2·604
1	2D	− ·00701†	+ ·9870†	+ ·9868†	+ ·0254†	− 1·3144†	− 1·3136†
e'	$2D+l'$	− ·00291†	− 1·1735†	− 1·1739†	+ ·0103†	+ 1·6697†	+ 1·6709†
e'	$2D-l'$	− ·03693†	+ 3·1571†	+ 3·1561†	+ ·1394†	− 4·3241†	− 4·3169†
e'^2	$2D+2l'$	− ·0042†	+ ·0826†	+ ·0818†	+ ·017	− ·239	− ·239
e'^2	$2D-2l'$	− ·1265†	+ 7·1584†	+ 7·1534†	+ ·500	− 9·928	− 9·892
e	$l-2D$	− ·08582†	− 1·4897†	+ 1·4944†	+ ·2220	+ 1·8274	− 1·8518
ee'	$l-2D+l'$	− ·2948†	− 4·321	+ 4·349	+ 7·383	+ 4·797	− 4·941
ee'	$l-2D-l'$	− ·0520†	+ 1·188	− 1·178	+ ·2751	− ·501	+ ·440
ee'^2	$l-2D+2l'$	− ·7197	− ·911	+ 9·21	+ 1·785	+ 9·01	− 9·67
ee'^2	$l-2D-2l'$	− ·0716	− ·03	+ ·05	− ·425	+ ·24	− ·31
e	$l+2D$	− ·0026	+ ·4972	+ ·4972	− ·0008	− ·6248	− ·6246
ee'	$l+2D+l'$	− ·0010	− ·732	− ·732	− ·0078	+ 1·1719	+ 1·1717
ee'	$l+2D-l'$	− ·0114	+ 1·795	+ 1·795	+ ·0340	− 2·7040	− 2·7038
e^2	$2l-2D$	+ ·0758†	+ ·6487†	− ·5969†	− ·2228	− ·8064	+ ·6010
e^2e'	$2l-2D+l'$	+ ·2552	+ 1·73	− 1·53	− ·663	− 1·81	+ ·93
e^2e'	$2l-2D-l'$	+ ·0785	− ·18	+ ·32	− ·400	− ·93	+ ·27
k^2	2F	+ 2·9901	− ·2001	− ·1717	− 2·9549	+ ·3785	+ ·2873
k^2e'	$2F+l'$	+ 2·8805	− ·25	− ·19	− 2·937	+ ·509	+ ·275
k^2e'	$2F-l'$	+ 3·0882	− ·37	− ·27	− 2·939	+ ·645	+ ·349
k^2	$2F-2D$	− ·2062†	+ 1·9756†	− 1·9690†	+ ·4637	− 1·9243	+ 1·9019
k^2e'	$2F-2D+l'$	− ·8051	+ 6·22	− 6·18	+ 1·967	− 6·282	+ 6·164
k^2e'	$2F-2D-l'$	− ·1653	− 2·31	+ 2·31	− ·319	+ 2·583	− 2·605

Values of M_i, j_1M_i, j_2M_i, j_3M_i—continued.

Char.	Angle.	j_2M_1	j_2M_2	j_2M_3	j_3M_1	j_3M_2	j_3M_3
1	0	+ ·00922	+ ·01223	0	− ·02415	+ ·00146	0
e'	l'	+ ·01105	+ ·02710	− ·00403			
e'^2	$2l'$	+ ·0154	+ ·0413	+ ·0032			
1	2D	− ·00057	− ·01583	− ·01583	+ ·0008	− ·0081	− ·0081
e'	$2D+l'$	− ·00014	+ ·0685	+ ·0685			
e'	$2D-l'$	− ·00235	− ·04494	− ·04490			
e'^2	$2D+2l'$	+ ·0010	+ ·0004	+ ·0004			
e'^2	$2D-2l'$	− ·0067	− ·0918	− ·0916			
e	l				+ ·0116		

Char.	Angle.	M_1	M_2	M_3	M_6	M_7	M_8	M_9
a_1	D	+ ·1132	+ ·3501	+ ·3681	+ ·972	+ 1·000	− ·007	− ·007
a_1e'	$D+l'$				+ ·380	+ ·435	− ·194	− ·195
a_1e'	$D-l'$				+ 2·505	+ 2·639	− ·029	− ·029
a_1e	$l-D$	+ ·00396	− ·31363	+ ·26736	− 1·464	+ 1·002	+ ·0139	− ·0088
a_1ee'	$l-D+l'$				− 3·53	+ 2·35	+ ·056	− ·033
a_1k^2	$3D-2F$	+ ·0951	+ ·977	+ ·983	− 1·57	− 1·69	+ ·980	+ ·980
a_1k$^2e'$	$3D-2F+l'$				+ ·62	+ ·62	− 3·48	− 3·48
a_1e^2	$D-2l$	− ·1387	+ ·179	− ·131				
a_1e^3	$3l-2D$	+ ·02080	− ·01282	+ ·06138				
a_1e^3	$4D-3l$	− ·00226	− ·02460	− ·02394				
$a_1$ek^2	$l-2F$	− 3·497	+ ·2952	− ·6828				
$a_1$ek^2	$4D-l-2F$	+ ·01108	+ ·01113	+ ·01169				
a_1k^4	$4D-4F$	·00	− ·0169	− ·0165				

Char.	Angle.	M_4	j_1M_4
k	$2w_1-w_3-h''$	+ 1·004	− 1·165
ke'	$2w_1-w_3-h''+l'$	+ ·896	− 1·021
ke'	$2w_1-w_3-h''-l'$	+ 1·118	− 1·340
k	w_3-h''	− 1·0040†	+ 1·1681†
ke'	$w_3-h''+l'$	− 1·087†	+ 1·343†
ke'	$w_3-h''-l'$	− ·932†	+ 1·030†
k	$2T-w_3-h''$	− ·0457	+ ·1213
ke'	$2T-w_3-h''+l'$	− ·1887	+ ·7669
ke'	$2T-w_3-h''-l'$	− ·0041	− ·0592
k	$2D+w_3-h''$	+ ·0355	− ·0857
ke'	$2D+w_3-h''+l'$	+ ·1421	− ·3682
ke'	$2D+w_3-h''-l'$	− ·0078	+ ·0227

Char.	Angle.	M_4
ek	$w_1+w_2-w_3-h''$	− 1·49
ek	$w_1-w_2+w_3-h''$	+ ·49
k^3	$-2w_1+3w_3-h''$	− ·989
e^{3}k	$w_1+3w_2-w_3-h''-2T$	− ·0154

CHAPTER XI.

THE DIRECT ACTION OF THE PLANETS.

Section (i). *The Disturbing Function.*

293. *Axes and Notation.*—The value of the disturbing function and the equations of variations are the same whatever axes be chosen. For the expansion of the former I take for plane of xy the ecliptic of 1850·0, and *for axis of x a line parallel to that joining the Earth and Sun*, that is, a line parallel to the Earth's *true* radius vector on the assumption that the Sun moves in an elliptic orbit. With these directions, x, y, z, r will now represent the coordinates and distance of the Moon, ξ, η, ζ, Δ those of the planet, the Earth being the origin.

For the elliptic coordinates of the Earth and planet with the Sun in the focus, I take r', r'' (the distances); V', V'' (the true longitudes); T′, T″ (the mean longitudes); ϖ', ϖ'' (the longitudes of the perihelia); 0, h'' (the longitudes of the nodes); 0, γ'' (the sines of half the inclinations; e', e'' (the eccentricities); $2a'$, $2a''$ (the major axes). All these longitudes are measured in the usual way, that is, from a *fixed* line in the plane of xy to the node, and then along the plane of the orbit.

The perturbations of the planet's orbit, like those of the Earth, are neglected in this chapter. In order that the motion of the Earth round the centre of mass of the Earth and Moon may be taken into account, the terms depending on a/a' in the disturbing function must be multiplied by the ratio of the difference of the masses of the Earth and Moon to their sum (Chap. I., § 4); we must therefore use $\alpha_1\mathrm{a}/a$ instead of a/a' (Chap. IX., § 193), or, with sufficient accuracy, α_1 instead of α.

The masses of a planet and of the Sun are denoted by m'', m', and the mean longitudes of the planets, measured like the other longitudes, as follows*: Mercury, Q; Venus, V; Earth, T; Mars, M; Jupiter, J; Saturn, S; the other planets and the asteroids will not be considered in this chapter.

294. *The Disturbing Function R and its Transformation.*—From Chap. I., Sect. (i), we obtain, on changing from the Sun to a planet as the disturbing body,

$$\frac{R}{m''}=\frac{1}{\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}^{\frac{1}{2}}}-\frac{x\xi+y\eta+z\zeta}{\Delta^3},\quad \Delta^2=\xi^2+\eta^2+\zeta^2 \quad . \quad . \quad . \qquad (1).$$

* Mean longitudes are denoted by roman capitals.

† *A.P.E.*, sect. ii., where the full details of the transformation will be found.

If we denote by $\partial/\partial Q$ the operator,

$$\frac{\partial}{\partial Q}=x\frac{\partial}{\partial \xi}+y\frac{\partial}{\partial \eta}+z\frac{\partial}{\partial \zeta},$$

the expansion of R may be put into the form

$$\frac{R}{m''}=\left(\frac{1}{2!}\frac{\partial^2}{\partial Q^2}-\frac{1}{3!}\frac{\partial^3}{\partial Q^3}+\ldots\right)\frac{1}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

or, since $1/\Delta$ is a solution of Laplace's equation, so that

$$\frac{\partial^2}{\partial \zeta^2}\frac{1}{\Delta}=-\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\frac{1}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

into the form

$$\frac{R}{m''}=\left[\tfrac{1}{4}(r^2-3z^2)\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)+\tfrac{1}{4}(x^2-y^2)\left(\frac{\partial^2}{\partial \xi^2}-\frac{\partial^2}{\partial \eta^2}\right)+xy\frac{\partial^2}{\partial \xi \partial \eta}+\left(x\frac{\partial}{\partial \xi}+y\frac{\partial}{\partial \eta}\right)z\frac{\partial}{\partial \zeta}+\text{terms of higher degree}\right]\frac{1}{\Delta} \quad (4),$$

Now, in polar coordinates, we have

$$\left.\begin{aligned}\xi&=-r'+(1-\gamma''^2)r''\cos(V''-V')+\gamma''^2 r''\cos(V''+V'-2h''),\\ \eta&=\phantom{-r'+{}}(1-\gamma''^2)r''\sin(V''-V')-\gamma''^2 r''\sin(V''+V'-2h''),\\ \zeta&=\phantom{-r'+{}}2\gamma''(1-\gamma''^2)^{\frac{1}{2}}r''\sin(V''-h'')\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5),$$

$$\Delta^2=\xi^2+\eta^2+\zeta^2=r''^2-r'^2-2r'\xi.$$

Hence $\partial/\partial\xi=-\partial/\partial r'$, and if f be any function of $\xi^2+\eta^2$ and ζ,

$$\frac{\partial f}{\partial V'}=-r'\frac{\partial f}{\partial \eta}-\xi\frac{\partial f}{\partial \eta}+\eta\frac{\partial f}{\partial \xi}=-r'\frac{\partial f}{\partial \eta},$$

$$\frac{1}{r'}\frac{\partial^2 f}{\partial V'^2}=\left(r'\frac{\partial}{\partial \eta}+\xi\frac{\partial}{\partial \eta}-\eta\frac{\partial}{\partial \xi}\right)\frac{\partial f}{\partial \eta}=r'\frac{\partial^2 f}{\partial \eta^2}+\frac{\partial}{\partial \eta}\left(\xi\frac{\partial f}{\partial \eta}-\eta\frac{\partial f}{\partial \xi}\right)+\frac{\partial f}{\partial \xi}=r'\frac{\partial^2 f}{\partial \eta^2}-\frac{\partial f}{\partial r'}.$$

Hence

$$\frac{\partial^2}{\partial \xi^2}=\frac{\partial^2}{\partial r'^2},\quad \frac{\partial^2 f}{\partial \eta^2}=\frac{1}{r'^2}\frac{\partial^2 f}{\partial V'^2}+\frac{1}{r'}\frac{\partial f}{\partial r'},\quad \frac{\partial^2 f}{\partial \xi \partial \eta}=\frac{\partial}{\partial r'}\left(\frac{1}{r'}\frac{\partial f}{\partial V'}\right) \quad . \quad . \quad . \quad . \quad (6).$$

Further, since $(\partial^2/\partial\xi^2+\partial^2/\partial\eta^2)f$ is a function of $\xi^2+\eta^2$, ζ, only,

$$\frac{\partial}{\partial \eta}\left(\frac{\partial^2 f}{\partial \xi^2}+\frac{\partial^2 f}{\partial \eta^2}\right)=-\frac{1}{r'}\frac{\partial}{\partial V'}\left(\frac{\partial^2 f}{\partial r'^2}+\frac{1}{r'^2}\frac{\partial^2 f}{\partial V'^2}+\frac{1}{r'}\frac{\partial f}{\partial r'}\right) \quad . \quad . \quad . \quad . \quad (7),$$

and since we may differentiate each of the equations (6) with respect to ξ, that is, to $-r'$, we can find all the derivatives of the third order with respect to ξ, η in terms of derivatives with respect to r', V'; and so on.

Again, by expressing Δ in polar coordinates, I have constructed the formula *

$$\frac{\partial}{\partial \zeta}\frac{1}{\Delta}=-\frac{\gamma''}{(1-\gamma''^2)^{\frac{1}{2}}}\frac{e^{\iota(V'-h'')}}{r'}\left\{\frac{\partial}{\partial V'}+\frac{1}{2\gamma''^2}\frac{\partial}{\partial h''}+\iota(1-\gamma''^2)\frac{\partial}{\partial \gamma''^2}\right\}\frac{1}{\Delta} \quad . \quad . \quad . \quad . \quad (8),$$

which may be combined with the previous equations, since the left member of (8) shows that the right-hand member is a function of $\xi^2+\eta^2$, ζ, and that its imaginary part is zero.

* *A.P.E.*, p. 19.

Finally

$$r'\frac{\partial}{\partial r'} = a'\frac{\partial}{\partial a'}, \quad \frac{\partial}{\partial V'} = \frac{\partial}{\partial \mathrm{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and therefore *if* $1/\Delta$ *be expanded in terms of the elliptic elements of the Earth and the planet, the functions needed are all expressed as derivatives of* $1/\Delta$ *with respect to the elements present in the development.* It is true that these derivatives are to be multiplied by $1/r'^2$ and that the lunar coordinates have to be transformed so as to be referred to the true instead of the mean place of the Sun; but the work needed to perform these two operations is very small, especially when compared with the labour of making developments of several different planetary functions, such as

$$\frac{1}{\Delta^3} - \frac{3\zeta^2}{\Delta^5}, \quad \frac{\xi^2 - \eta^2}{\Delta^5}, \quad \frac{\xi\zeta}{\Delta^5}, \text{ etc.}$$

295. *The transformed Disturbing Function.*—I omit the algebraical details necessary to follow out this method, so as to present the results in a form convenient for numerical application. The result is a form for the disturbing function expressed as a sum of products: the first factor in each product is a function of the Moon's coordinates, u, s, z, multiplied by a certain function of r', V'; the second factor consists of derivatives of $1/\Delta$ with respect to a', T, h'', γ''^2.

The notation for the first factors has already been given in § 285 of the last chapter. If an angle θ is present in the first factor, then the corresponding term is $M_i \cos\theta + \iota M_i \sin\theta$, where such constant factors have been taken out that the M_i may be numerical quantities.

296. For the second factors, let a term with argument ϕ in $1/\Delta$ be

$$\frac{1}{\Delta} = P\cos\phi.$$

If i, i' be the multiples of T, $2h''$ present in ϕ, the operators $\partial/\partial V'$, $\partial/\partial h''$ give rise to the factors i, i', and the cosine is changed to a minus sine.

Also P is of degree -1 in length, and may be expanded in the form $f(\alpha)/a'$, where $\alpha = a'/a''$, or in the form $f(\alpha)/a''$, where $\alpha = a''/a'$, according as the orbit of the disturbing planet is outside or inside that of the Earth. If, then, we put $I = \alpha d/d\alpha$ for a derivative with respect to $\log\alpha$ only in so far as α occurs explicitly in P after P has been expressed in one or other of these two forms, we have:

$$\left.\begin{array}{llll} \text{For outer planets,} & r'\dfrac{d}{dr'} = \alpha\dfrac{d}{d\alpha} = I, & \alpha = \dfrac{a'}{a''}, & P = \dfrac{1}{a''}f(\alpha); \\ \text{For inner planets,} & r'\dfrac{d}{dr'} = -\alpha\dfrac{d}{d\alpha} - 1 = -I - 1, & \alpha = \dfrac{a''}{a'}, & P = \dfrac{1}{a'}f(\alpha) \end{array}\right\} \quad . \quad . \quad . \quad (11).$$

When we combine the two factors, we obtain the product of two cosines or two sines multiplied by a constant factor; this product is expressed as the cosine of the sum or difference of the two angles θ, ϕ. The notation for the planet coefficients is as follows:—

Let J_+, J_- denote the operators

$$(1-\gamma''^2)\frac{\partial}{\partial\gamma''^2} \pm \left(i + \frac{i'}{\gamma''^2}\right),$$

and let i, i' denote the multiples of T, $2h''$ present in ϕ.

For *outer* planets put

$$P_1 = (I^2 - i^2)P, \quad P_2 = \tfrac{1}{2}P_1 - IP + i^2P, \quad P_3 = -i(I-1)P, \quad P_4 \pm P_5 = -(I \pm i)J_{\pm}P \quad . \quad . \quad (12),$$

$$\left.\begin{aligned} &P_6 \pm P_7 = (I - 2 \mp i)P_1, \quad P_8 \pm P_9 = (I \mp 3i - 6)P_1 + 4(1 \pm i)(2 \pm i)(I \mp i)P, \\ &P_{10} \pm P_{11} = J_{\pm}(-P_1 + 2IP \pm 2iP), \quad P_{12} \pm P_{13} = -J_{\pm}\{P_1 + (\pm 2i - 2)IP + (2i^2 \mp 2i)P\} \end{aligned}\right\} \quad . \quad (13),$$

all the upper or all the lower signs in any equation being taken together.

The product of the pairs of factors gives for any term of argument $\theta \pm \phi$ in R,

$$R = \frac{m''}{4m'}n'^2\mathrm{a}^2 \cdot aa''\Big[M_1P_1 + M_2P_2 \mp M_3P_3 - \frac{2\gamma''}{(1-\gamma''^2)^{\frac{1}{2}}}M_4(P_4 \pm P_5)$$

$$+ \tfrac{1}{4}a_1(M_6P_6 \mp M_7P_7 + M_8P_8 \mp M_9P_9) - \frac{\gamma''a_1}{2(1-\gamma''^2)^{\frac{1}{2}}}\{M_{10}(P_{10} \pm P_{11}) + M_{12}(P_{12} \pm P_{13})\}\Big]\cos(\theta \pm \phi) \quad . \quad (14),$$

all the upper or all the lower signs being taken according as it is convenient to use $\theta + \phi$ or $\theta - \phi$.

For *inner* planets, we replace aa'' by a' in this formula, and I by $-I-1$ in (12), (13).

It is to be noticed that P_ia'' for outer planets and P_ia' for inner planets are numerical coefficients when the value of a has been substituted.

297. *Method of computation of the coefficients in R.*—The terms in R with suffixes 1, 2, 3 give rise to nearly all the sensible perturbations of the Moon's orbit. In the great majority of cases $M_2P_2 \mp M_3P_3$ is small compared with either of the two terms, and this is due to the approximate numerical equality of M_2 and M_3 and of P_2 and P_3. It is therefore better (and the computations were so made) to use these two terms in the form

$$\tfrac{1}{2}(M_2 \mp M_3)(P_2 + P_3) + \tfrac{1}{2}(M_2 \pm M_3)(P_2 - P_3);$$

and then again one of these two expressions was generally small compared with the other. Another advantage of this form arose from the fact that it was sufficient to have M_2, M_3 to four significant figures and P_2, P_3 to six, instead of both to six. And, moreover, the near equality of P_2, P_3 can be foretold by the theory, when it exists. A similar circumstance holds with the pairs with suffixes 4, 5; 6, 7; 8, 9; 10, 11; 12, 13.

There is not a large number of terms depending on the terms in R with suffixes 4, 5, fewer still with 6, 7, 8, 9, and none sensible with 10, 11, 12, 13. It did not seem necessary to carry the computations to the next term of R, depending on a_1^2; this coefficient gives a factor 6×10^{-6} compared with the first terms of R, and it gives rise to no arguments which are not present in the first terms.

Section (ii). *The Computation of the Coefficients* P_i.*

298. *Leverrier's Expansion of* $1/\Delta$.†—LEVERRIER's literal expansion in powers of the eccentricities and mutual inclinations, with coefficients depending on a and arguments on T, ϖ', ϖ'', h'', was used. Here the Earth is supposed to move in the plane of reference, and the notation is slightly different. I have, therefore, put in the development—

For LEVERRIER's symbols,	τ, τ', l', λ, ω, e, a, η
the symbols	h'', h'', T, T'',‡ ϖ'', e'', a'', γ''

for inner planets, and have then interchanged the accents of a, ϖ, e, T for outer planets.

LEVERRIER's development contains functions of a through certain coefficients $\beta^{(i)}_{s,p}$, which are defined as follows. Put

$$\left.\begin{aligned} &\Delta_0^2 = 1 + a^2 - 2a\cos(\mathrm{T}-\mathrm{T}''), \\ &\frac{a}{\Delta_0^{s}} = \tfrac{1}{2}a'\sum_{i=-\infty}^{i=+\infty}\beta_s^{(i)}\cos i(\mathrm{T}-\mathrm{T}''), \qquad \beta^{(i)}_{s,p} = \frac{1}{p!}a^p\frac{d^p}{da^p}\beta^{(i)}_s \end{aligned}\right\} \quad . \quad . \quad . \quad (15).$$

Instead of β_s, he uses the letters A, B, C, D, $\mathcal{H}$, $\mathcal{A}$,§ according as $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$. I adopted the same notation, as well as the following:

$$E^{(i)} = \tfrac{1}{2}(B^{(i-1)} + B^{(i+1)}), \qquad G^{(i)} = \tfrac{3}{8}(C^{(i-2)} + 4C^{(i)} + C^{(i+2)}),$$
$$H^{(i)} = \tfrac{5}{16}(D^{(i-3)} + 9D^{(i-1)} + 9D^{(i+1)} + D^{(i+3)}), \qquad L^{(i)} = \tfrac{3}{4}(C^{(i-2)} + C^{(i)}),$$
$$S^{(i)} = \tfrac{15}{16}(D^{(i-3)} + 3D^{(i-1)} + D^{(i+1)}), \qquad T^{(i)} = \tfrac{15}{16}(D^{(i-3)} + D^{(i-1)}),$$

to which the suffix p may be attached according to the previous definition. But I dropped the brackets round the indexes to the letters, since powers of these functions do not arise; and the indexes themselves were also dropped whenever they were all the same, equal to i, in a given equation.

299. *Formulæ for computing the coefficients.*—LEVERRIER gives the numerical values in most cases up to $i = 10$, but they are needed much further in many cases of the lunar problem. Hence, all of them were newly computed by the formulæ which are fully set forth in Section III. of my Adams Prize Essay. These formulæ are constructed for several purposes: first, for finding isolated coefficients for special values of i; second, for making tables of coefficients for many consecutive values of i; third, for the avoidance of those small coefficients which appear as the difference of two large numbers. The third point is a difficulty which arises chiefly in computing $(I^2 - i^2)\beta^i_{s,p}$, the two parts of which are large compared with their difference, especially for large values of i; but the formulæ for these completely surmounted the difficulty.

* *A.P.E.*, sect. iii.

† *Ann. Obs. Paris* (*Mém.*), vol. i., where the expansion is given so as to include terms of the seventh order with respect to e', e'', γ''. BOQUET (*ib.*, vol. xix) has computed the terms of the eighth order.

‡ Denoted P in *A.P.E.*, by inadvertence the same letter as for the general coefficient.

§ LEVERRIER does not need the last two.

It arises in a less troublesome form in P_2, P_3, but there the number of places of decimals computed was always sufficient for the degree of accuracy required.

300. *Numerical values of A^i_p, B^i_p. . .*—These were computed to six significant figures in tables * as follows:—

Venus: A^i_0, B^i_0 to $i=43$; A^i_p, B^i_p to $p=4$, $i=30$; C^i_p to $p=2$, $i=30$;
D_0, D_1, $\mathfrak{H}_0$ to $i=30$; $\{(I+1)^2-i^2\}A^i_p$ to $p=3$, $i=30$;
$\{(I+1)^2-(i+1)^2\}B^i_p$ to $p=2$, from $i=-30$ to $i=30$; $\{(I+1)^2-i^2\}C^i_p$ to $p=1$, $i=30$.
Jupiter to $i=6$ for A^i_p to $p=3$, B^i_p to $p=3$, C^i_p to $p=1$, D^i_0.
Mars: A^i_0, B^i_0 to $i=30$; and to $i=6$ for A^i_1, A^i_2, B^i_1, B^i_2, C^i_0.
Mercury to $i=8$ for A^i_p, B^i_p, to $p=4$, C^i_p to $p=2$, D^i_0.

All other coefficients required with these planets, and those for Saturn, were separately computed as the needs for them arose.

Section (iii). *The Sieve.**

301. The larger number of the terms in R which give rise to sensible coefficients in the coordinates have periods which are comparable with the month or the year. For such terms the obvious plan was to take the successive values of θ (the argument arising from the lunar factors) according to the magnitudes of the coefficients which accompanied this factor. The arguments 0, l, $2\mathrm{D}-l$, $2\mathrm{D}$, . . . were successively combined with all the possible arguments ϕ until the terms became insensible and it was unnecessary to proceed further. In each case the argument ϕ was divided into the series $i(\mathrm{T}-\mathrm{T}'')$, $i(\mathrm{T}-\mathrm{T}'')\pm\varpi'$, and so on, the magnitude of each series mainly depending on the power of e', e'', γ'' which accompanied it; and again each of these series was computed with a sufficient number of values of i with each Moon argument. A little practice quickly enabled one to choose out the largest coefficient in each set, and a rough calculation was sufficient to show whether the term would be sensible. This rough calculation had to be made for both δw_1, δw_2, and sometimes δw_3, for, with terms of short period, the secondary inequalities (§ 282) are frequently sensible when the primary might be neglected.

This method could not fail as long as the periods were short. These periods only arise in the equations of variations (§ 269) through the divisors s, s^2. But if any period was long, then s, s^2 would be comparatively small, and a large coefficient might result. Such cases as occurred during the progress of the calculations were naturally dealt with as they arose; the only matter which called for attention was the necessity for a larger number of significant figures in the coefficients M_i, P_i. If, in the general method, the numbers were not sufficiently accurate, the special coefficient was separately computed again with more places of decimals.

But the lunar terms contain multiples of four different arguments, $w_1-\mathrm{T}$, w_2, w_3, $\mathrm{T}-\varpi'$, that is, combinations of four different periods, and the planet terms two periods, T'', T. Hence we may have combinations of five different periods, and there will be

* *A.P.E.*, sect. v., where the numerical values will be found.

* *A.P.E.*, sect. iv.

many long periods arising therefrom. It was therefore necessary to sift out those which would give sensible coefficients.

302. *The Sieve.*—The method was essentially the same as that for the short-period terms, but, as there were thousands of possible combinations, some plan had to be devised to find an upper limit to each coefficient so rapidly that every coefficient might be examined within a reasonable time.

The limitations were as follows. Only primary terms were examined. It was shown* that the secondary would not be greater than the primary unless $i_1 = 0$ (§ 279), $s > 60''$ or $i_1 = 1$, $s > 1000''$; the very few of the former were separately examined, and the latter had been treated in the short-period terms. Periods greater than 3500 years ($s < 1''$) or coefficients $< 0''\cdot 01$ were to be excluded; neither could sensibly affect the motion of the Moon within historic times. But one or two longer periods with coefficients greater than $0''\cdot 01$ which appeared in the course of the work were retained.

The possible long periods were then constructed by finding all up to the largest values of i, j in $i(\mathrm{T} - \mathrm{T}'') + j\mathrm{T}$, $\pm w_1 + i(\mathrm{T} - \mathrm{T}'') + j\mathrm{T}, \ldots$ It was soon seen that only a dozen or so in each set need be retained, and at the most three multiples of w_1. Then a table was formed for the multiples of w_2, w_3, giving the periods, the lowest orders with reference to e, k which would accompany each multiple, and the multiple of T which would occur in the lunar argument with this lowest order. Thus, for a given multiple of w_2, w_3, the various long-period combinations with the former sets could be seen at a glance.

303. Next, very simple formulæ were constructed for the primary coefficient, depending only on the power of e, k present in the lunar factor, the multiple of $\mathrm{T} - \mathrm{T}''$ and the coefficient present in the expansion of $1/\Delta^3$ or of $1/\Delta^5$. These formulæ arose from transformations of the disturbing function somewhat similar to those of § 294, but depending on derivatives with respect to T only. The values of the coefficients in the expansions of Δ^{-3}, Δ^{-5} were obtained from NEWCOMB's table† of these coefficients in the case of Venus; partly from his incomplete table,‡ and partly by extrapolation and by approximate computations, in the case of Mars; for the other planets, which presented little difficulty, a table for $\Delta_0^{\ 3}$ was roughly computed, and simple formulæ depending also on the order of the eccentricities and inclination were constructed.

The various coefficients were examined according as they arose from terms whose characteristics were of orders 0, 1, 2, ... With each order was associated a maximum value of s which could give sensible coefficients; after the first three or four orders the work went very rapidly, as these maximum values of s became small, and the great majority of the terms could be excluded without computation.

About 100 long-period terms were retained out of several thousand examined, and their coefficients were accurately computed. In no case did these coefficients exceed the preliminary estimates found by means of the sieve. No new terms of any great

* *A.P.E.*, p. 38. † *Wash. Astr. Papers*, vol. v., pp. 248–257. ‡ *L.c.*, pp. 258–261.

importance were found, and the corrections to RADAU'S values* of those previously computed were small from the observational standpoint, as far as the long-period primaries were concerned.

Section (iv). *Numerical Values of the Elements.*

304. Most of the observed quantities required are known with more than sufficient accuracy. The most doubtful is the mass of Mercury, which may be in error by as much as 50 per cent.; but the largest coefficient with the adopted value is less than $0''\cdot08$, and the term has a period of 39 years. The mass of Venus may be in error by 1 or 2 per cent., giving a maximum possible error in the largest coefficient (period 273 years) of $0''\cdot3$ from this cause.

The values of the elements used are shown in the following tables:—

Arg.	Daily motions of arguments.	Epoch 1850·0.	Longitudes at Epoch. Perigee.	Longitudes at Epoch. Node.
w_1	47434·891″	Mercury	75° 07′ 19″	46° 33′ 12″
w_2	400·923	Venus	129° 27′ 34″	75° 19′ 47″
w_3	− 190·772	Earth	100° 21′ 40″	
Q	14732·420	Mars	333° 17′ 55″	48° 24′ 01″
V	5767·670	Jupiter	11° 54′ 27″	98° 55′ 58″
T	3548·193	Saturn	90° 06′ 40″	112° 20′ 51″
M	1886·518			
J	299·129			
S	120·455			

	Eccentricity.	Inclination.	Sine half inclin.	$\log \frac{a''}{a'}$.	$\frac{m'}{m''}$.
Moon $e =$	·10955		$k =$ ·044780	$\log a_1 = \bar{3}\cdot3988$	
,, $e =$	·054906		$\gamma =$ ·044887		
Earth	·016772				
Mercury	·205604	7° 00′ 07″	·061066	$\bar{1}\cdot5878216$	6000000
Venus	·0068446	3° 23′ 35″·3	·0296063	$\bar{1}\cdot8593374$	408000
Mars	·093261	1° 51′ 02″	·016149	·1828960	3093500
Jupiter	·048254	1° 18′ 42″	·011466	·7162374	1047·35
Saturn	·056061	2° 29′ 39″	·022	·9794957	3501·6

* *L.c.* (§ 279), p. 113.

Section (v). *The Final Results.*

305. The detailed results arising from each term of the disturbing function are fully set forth in *A.P.E.*, Section vi., and they will not be reprinted. Many of the resulting terms in the Moon's coordinates, especially those arising from the short-period terms in R, have the same arguments, and must be combined. The final results only will be given here.

There are two methods of expressing the perturbations. The first is to add them to the true longitude, latitude, and parallax of the Moon; the second is to leave them as additions to the elements w_1, w_2, w_3, a, e, γ which would be tabulated with these additions. This latter method is only of special advantage for tabular purposes when the variations of a, e, γ may be neglected, and this happens only with terms of long period in which w_1 is present. If w_1 is absent from the primary, and the period of the term is not very long compared with the periods of the Moon's node or perigee, the variation of e produces an effect of the same order as the variation of w_2, and that of γ as w_3; in fact, the statement in § 282 has to be remembered. In these cases the variations of a (or n) are insensible, and it may be convenient to retain the variations of w_1 (or of w_1, w_2) as elemental inequalities, adding the parts due to the other elements to the coordinates. No periodic variations of e have been retained as elemental inequalities.

For certain other cases in which w_3 is present, with δe, δa insensible, it is best to retain δw_1, δw_2, δw_3 as elemental terms, and account for those arising from $\delta\gamma$ by multiplying the final value of the latitude by the variable factor $1+\delta\gamma/\gamma$, and the terms in longitude containing the argument 2F by $1+2\delta\gamma/\gamma$. But in setting forth the results such terms are left as perturbations of γ. Hence the terms are placed in two classes, those added to the coordinates, and, in addition, those added to the elements.

The original limit set was $0''{\cdot}01$, but all short-period terms and most of the long-period terms have been computed to $0''{\cdot}001$, and they are so retained here. A star replaces the last figure in the cases where the computations were only made to $0''{\cdot}01$.

306. The tables are arranged according to the lunar arguments so that θ remains the same until a new value is set down, and then according to the multiple j of T (or j'' of T''), which again remains the same until a new value is set down, and finally, according to multiples of T − T''. The coefficients are set down in units of $0''{\cdot}001$, the angle α being so chosen ($<360°$) that they are all positive.

The value of the angle α is also not generally repeated when it is the same for a long series of terms.

307. $\delta V = +0''\cdot001\, C \sin\{\theta + jT + i(T - V) + \alpha^\circ\}$, *Venus.*

$\theta = 0$

j	i	α	C
0	1	0·0	480
	2		200
	3		92
	4		60
	5		38
	6		25
	7		17
	8		12
	9		8
	10		6
	11		4
	12		1
	21		3
1	−3	92·2	1
	−2		4
	−1		8
	1		47
	2	272·2	76
	3		21
	4		12
	5		7
	6		6
	7		4
	8		1
2	−18	209	50
	−4	27	1
	−3		2
	−2		2
	−1		3
	1		6
	2		8
	3		37
	4	207	8
	5		3
	6		4
	7		1
3	5	112	7

$\theta = 2D$

j	i	α	C
0	−15	0	1
	−14		2
	−13		2
	−12		2
	−11		3
	−10		5
	−9		6
	−8		8
	−7		8
	−6		11
	−5		11
	−4		10
	−3	180	36
	−2	0	26
	−1		15
	1		15
	2		8
	3		4
	4		4
	5		4
	6		3
	7		3
	8		3
	9		2
	10		1
	18		3
1	20	273	3
−1	−15	78	1
	−14		1
	−13		2
	−12		2
	−11		3
	−10		4
	−9		4
	−8		4
	−7		5
	−6		5

$\theta = 2D$

j	i	α	C
−1	−5	84	7
	−4	78	7
	−3		4
	−2		4
	−1		3
	1		1
−2	−6	162	6
	15	151	4
	18	151	10

$\theta = l$

j	i	α	C
0	−8	180·0	2
	−7		4
	−6		5
	−5		6
	−4		9
	−3		16
	−2		29
	−1		68
	1	0·0	91
	2		64
	3	180·0	127
	4		7
	5		1
	22		2
1	1	92	8
	2	272	13
	3		6
	4		8
	5	92	4
	6		2
	23	272	6
−1	−5	268	1
	−4		2
	−3		5
	−2		13
	−1	88	8

$$\delta V = +0''\cdot 001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T}-\mathrm{V}) + \alpha^\circ\},\ \textit{Venus}.$$

$\theta = l$

j	i	α	C
−1	3	280	7
2	−18	209	3
	2	27	1
	3		6
	4	207	1
	6	198	16
−2	−4	333	1
	−3	153	6
	−2		1

$\theta = 2\mathrm{D} - l$

j	i	α	C
0	−13	180·0	1
	−12		2
	−11		2
	−10		3
	−9		6
	−8		8
	−7		13
	−6		22
	−5		39
	−4		87
	−3		716
	−2	0·0	152
	−1		74
	1		13
	2		10
	3		7
	4		5
	5		3
	6		2
	18		11
	19		2
1	−1	310	2
−1	−10	269	1
	−9		2
	−8		3
	−7		5
−1	−6	269	8
	−5		25
	−4	89	33
	−3		10
	−2		5
	−1		3
	17	253	3
−2	−7	340	3
	−6	162·5	83
	−5	165	4
	15	151	25

$\theta = 2\mathrm{D} + l$

j	i	α	C
0	−9	0	1
	−8		2
	−7		2
	−6		2
	−5		3
	−4		1
	−3		4
	−2		4
	−1		3
	1		1

$\theta = 2l$

j	i	α	C
0	−3	180	1
	−2		2
	−1		5
	1	0	5
	2		2
	3	180	9

$\theta = 2l - 2\mathrm{D}$

j	i	α	C
0	1	180·0	4
	2		11
	3		76
	4	0	3
1	4	92	4
2	6	17·5	65

$\theta = 2l - 4\mathrm{D}$

j	i	α	C
0	3	0	8
2	6	18	2

$\theta = -4\mathrm{D}$

j	i	α	C
0	3	0	7

$\theta = 3l - 2\mathrm{D}$

j	i	α	C
0	3	180	3
2	6	18	2

$\theta = 2\mathrm{F} - 2\mathrm{D}$

j	i	α	C
0	3	180	2

$\theta = \mathrm{D}$

j	i	α	C
0	20	0	2
1	3	273	5
−1	−3	87	27

$\theta = l - \mathrm{D}$

j	i	α	C
1	3	93	40

$\theta = l + \mathrm{D}$

j	i	α	C
−1	−3	87	2

$\theta = w_3$

j	i	α	C
−2	−5	255	3
	−4		9
	−3	75	16
	−2		5

$\theta = w_3 + 2\mathrm{F}$

j	i	α	C
−2	−3	75	3

308. $\delta V = +0''{\cdot}002 \sin(2D + T - 3Q + 105^\circ)$, *Mercury.*

309. $\delta V = +0''{\cdot}001\, C \sin\{\theta + j''M + i(M - T) + \alpha^\circ\}$, *Mars.*

$\theta = 0$

j''	i	α	C
0	1	0	19
	2		8
	3		3
	4		2
	5		1
1	0	212·7	5
	1		60
	2	33	10
	3		4
	4		2
	5		1
2	1	243	2
	2		12
	3	63	5
	4		2
	5		1
3	3	276	2
	4	96	1

$\theta = 2D$

j''	i	α	C
−1	−5	149	3
−2	−6	297	2

$\theta = l$

j''	i	α	C
0	−1	180	2
	1	0	2
	4	180	3
1	1	212	7
	2	32	1
	5	212	8
−1	−2	147	1
	−1	327	7
2	2	243	1
	6	63	6
−2	−2	297	1

$\theta = 2D - l$

j''	i	α	C
0	−6	180	1

$\theta = 2D - l$

j''	i	α	C
0	−5	180	3
	−4		17
	−3	0	5
	−2		2
−1	−7	149	1
	−6		3
	−5		38
	−4	329	3
	−3		1
−2	−6	297	30

$\theta = 2l - 2D$

j''	i	α	C
0	4	180	2
1	5	211	15
2	6	243	19

$\theta = 3l - 2D$

j''	i	α	C
2	6	243	1

310. $\delta V = +0''{\cdot}001\, C \sin\{\theta + j''J + i(J - T) + \alpha^\circ\}$, *Jupiter.*

$\theta = 0$

j''	i	α	C
0	1	0·0	69
	2	180	13
1	0	173·8	209
	1	354	11
	2		8
2	0	162	9

$\theta = 2D$

j''	i	α	C
0	−5	0	1
	−4		2
	−3		3

$\theta = 2D$

j''	i	α	C
0	−2	180·0	45
	−1	0	2
	2	180	2
1	0	174	2
−1	−2	7	20
	0		2

$\theta = l$

j''	i	α	C
0	−3	0	1
	−2		4
	−1		8

$\theta = l$

j''	i	α	C
0	1	180·0	8
	2		171
	3	15	2
1	0	173	21
	1	353·3	2
	2		58
	3	173	1
−1	−1	186	2
	0	6	21
	3	286	8
2	2	342	2

$$\delta V = +0''\cdot 001\, C \sin\{\theta + j''\mathrm{J} + i(\mathrm{J}-\mathrm{T}) + \alpha^\circ\}, \textit{ Jupiter.}$$

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−4	180	4
	−3	183	20
	−2	180·0	804
	−1	0	7
1	−3	260	4
	−2	353	7
−1	−4	187	1
	−3	187	6
	−2	6·7	306
	−1	280	5
−2	−2	18	9
	−1	107	2

$\theta = 2\mathrm{D} + l$

j''	i	α	C
0	−2	180	3
	2	180	1
−1	−2	7	1

$\theta = 4\mathrm{D} - l$

j''	i	α	C
0	−2	180	7
−1	−2	7	3

$\theta = 2l$

j''	i	α	C
0	2	180	13
1	2	353	2

$\theta = 2l - 2\mathrm{D}$

j''	i	α	C
0	2	180·0	187
1	2	173·3	190
2	2	162	2

$\theta = 2l - 4\mathrm{D}$

j''	i	α	C
0	2	0	9
1	2	173	6

$\theta = 3l - 2\mathrm{D}$

j''	i	α	C
0	2	180	7
1	2	173	5

$\theta = 2\mathrm{F} - 2\mathrm{D}$

j''	i	α	C
0	2	180	2

$\theta = 2\mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	2	180	2

$\theta = -2\mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	2	0	1

$\theta = w_3$

j''	i	α	C
1	0	81	4

311. $\delta V = +0''\cdot 001\, C \sin\{\theta + j''\mathrm{S} + i(\mathrm{S}-\mathrm{T}) + \alpha^\circ\}$, *Saturn.*

$\theta = 0$

j''	i	α	C
1	0	90	24

$\theta = l$

j''	i	α	C
0	2	180	3
1	0	90	3
−1	0	90	3

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−2	180	14
−1	−2	270	4

312. $\delta V = +0''\cdot 001\, C \sin\psi$, *all planets.*

ψ	C	ψ	C
$l' + 198^\circ$	14	$l - l' + 168^\circ$	2
$2l' + 228^\circ$	4	$2\mathrm{D} - l$	39
$2\mathrm{D}$	17	$2\mathrm{D} - l - l' + 168^\circ$	1
$2\mathrm{D} - l' + 338^\circ$	2	$2\mathrm{D} + l$	2
$l + l' + 192^\circ$	2	$2\mathrm{D} - 2l$	2

313. $\delta U = +0''\cdot001 \sin\{\theta + jT + i(T - V) + \alpha^\circ\}$, *Venus.*

$\theta = \pm F$

j	i	α	C
0	1	0	5
	2		3
	3		1
1	5	90	2
2	−18	209	2

$\theta = F + 2D$

j	i	α	C
0	−4	0	2
	−3	180	1
	−2	0	3
	−1		3

$\theta = -F + 2D$

j	i	α	C
0	−8	180	2
	−7		3
	−6		5
	−5		9
	−4		23
	−3	0·0	45
	−2		14
	−1		9
−1	−6	90	2
	−5	270·0	68
	−4	90	2

$\theta = \pm F - l$

j	i	α	C
0	1	0	2
	2		1

$\theta = \pm F + l$

j	i	α	C
0	1	0	2
	2		1
	3	180	6

$\theta = F + l - 2D$

j	i	α	C
0	1	180	2
	2		6
	3	0	34
	4		4
	5		2
1	5	90	4
2	6	18	4

$\theta = -F + l - 2D$

j	i	α	C
0	1	180	2
	2		6
	3	0	32
	4		4
	5		2
2	6	18	4

$\theta = F - l - 2D$

j	i	α	C
0	3	180	2
1	5	270	4

$\theta = \pm F + 2l - 2D$

j	i	α	C
0	3	180	3
2	6	18	3

$\theta = w_1$

j	i	α	C
0	−7	285	2
	−6		3
	−5		5
	−4		6
	−3		9
	−2		14
	−1		27
	1	105	15
	2		6
	3		3
−2	−8	255	2
	−7		3
	−6		5
	−5		9
	−4		25
	−3	75·3	72
	−2		18
	−1		10
	1		6
	2		4
	3		3

$\theta = w_1 - 2D$

j	i	α	C
−2	−3	255	2

$\theta = w_1 \pm l$

j	i	α	C
−2	−3	75	±4

314. $\delta U = +0''\cdot001\, C \sin\psi$, *Jupiter and Mars.*

ψ	C
$\pm F + J + 348^\circ$	2
$F - 2D + 2J - 2T + 180^\circ$	23
$F - 2D + 3J - 3T$	5
$-F - 2D + 2J - 2T$	3
$\pm F + l + 2J - 2T + 180^\circ$	7
$\pm F + l + 3J - 2T + 353^\circ$	3
$\pm F + l - 2D + 2J - 2T$	36
$\pm F + l - 2D + 3J - 2T + 173^\circ$	14
$\pm F + 2l - 2D + 2J - 2T + 180^\circ$	8
$\pm F + 2l - 2D + 3J - 2T + 173^\circ$	9
$w_1 + 2M - T + 345^\circ$	10
$w_1 + T - J + 81^\circ$	2
$w_1 + J + 69^\circ$	5
$F + 2D$	2
$-F + 2D$	5
$w_1 - 2T + 75^\circ$	8

315. δ (*Parallax*) $= +0''\cdot001\,C \cos\psi$.

ψ	C
$l-2D+3T-3V$	6
$l-2D+2J-2T$	7
$l-2D+3J-2T+173^\circ$	3

316. *Terms added to the elements.*

$$\delta w_1 = +0''\cdot001\,C \sin\psi.$$

ψ	C	ψ	C
$13T-8V+321^\circ$	3	$F+24T-23V+285^\circ$	3
$l+3T-10V+33^\circ$	35*	$D-l+F+20(T-V)+166^\circ$	2
$l+16T-18V+151^\circ\cdot0$	1455*	$D+l-F+17T-18V+75^\circ$	8
$l+29T-26V+112^\circ\cdot0$	108	$3D-3l+F+25T-22V+134^\circ$	2*
$l+21(T-V)$	30		
$2D-l+21T-20V+273^\circ\cdot0$	126	$2D-l+5T-4Q+113^\circ$	3
$2D-l+8T-12V+303^\circ$	33	$2D-l+T-3Q+105^\circ$	75
$2F-2D+6T-5V+270^\circ$	54	$2F-l+3T-4Q+67^\circ$	3
$3l-2D+24(T-V)$	10	$3D-F-l+2T-3Q+47^\circ$	2
$l+2F-4D-15(T-V)$	2		
$D+12T-15V+262^\circ$	13	$4D-3l+25M-23T+67^\circ$	4*
$D+25T-23V+190^\circ$	13	$D-F+2M+165^\circ$	17
$3D-2F+19T-18V+272^\circ$	2	w_3+110°	7

$$\delta w_2 = +2''\cdot69\ (\text{No. of years from } 1850\cdot0)$$
$$+0''\cdot118 \sin(l+16T-18V+331^\circ\cdot0)$$

$$\delta w_3 = -1''\cdot42\ (\text{No. of years from } 1850\cdot0)$$
$$+1''\cdot86 \sin(w_3+290^\circ\cdot1) + 0''\cdot172 \sin(l+16T-18V+151^\circ\cdot0)$$

$$\delta\gamma = +0''\cdot083 \cos(w_3+110^\circ\cdot1)$$

CHAPTER XII.

THE INDIRECT ACTION OF THE PLANETS.

Section (i). *The Disturbing Function.*

317. *Transformation to coordinates used in the direct action.*—The disturbing function for the action of the Sun on the Moon is (Chap. I., § 3)

$$\frac{m'}{r'^3}\left\{\frac{r^2-3z^2}{4}+\tfrac{3}{4}\rho^2\cos 2(V-V')+\tfrac{5}{8}\frac{\rho^3}{r'}\cos 3(V-V')+\tfrac{3}{8}\frac{(r^2-5z^2)}{r'}\rho\cos(V-V')\right\} \quad . \quad . \quad (1),$$

to a sufficient approximation. Let $\delta r'$, $\delta V'$ be the perturbations of r', V' from elliptic motion, the plane of reference being the same as before. Put $\delta r'/r'=\delta\rho'$ and neglect powers of $\delta\rho'$, $\delta V'$ beyond the first. Then the disturbing function due to $\delta\rho'$, $\delta V'$ is

$$R=\frac{3m'}{4r'^3}\left[-\delta\rho'\{r^2-3z^2+3\rho^2\cos 2(V-V')\}+\delta V'\{2\rho^2\sin 2(V-V')\}\right]$$
$$+\frac{3m'}{2r'^4}\left[-\delta\rho'\{\tfrac{5}{3}\rho^3\cos 3(V-V')+(r^2-5z^2)\rho\cos(V-V')\}+\delta V'\{\tfrac{5}{4}\rho^3\sin 3(V-V')+\tfrac{1}{4}(r^2-5z^2)\rho\sin(V-V')\}\right] (2)$$

Replace the functions of the coordinates of the Moon and Sun by the expressions given in Chap. X., § 285, so that R will now denote that part of the disturbing function which depends on the lunar angle θ. We obtain

$$R=\frac{3m'}{2}\frac{\mathrm{a}^2}{a'^3}\frac{a'}{r'}\left[-\delta\rho'(M_1+\tfrac{3}{2}M_2)\cos\theta+\delta V' M_3\sin\theta-\delta\rho'\frac{\mathrm{a}}{a'}(M_6+5M_8)\cos\theta+\delta V'\frac{\mathrm{a}}{a'}(\tfrac{1}{4}M_7+\tfrac{15}{4}M_9)\right] \quad (3).$$

We can therefore obtain the required lunar functions directly from the results given in § 292 if we multiply all the series there tabulated by a'/r'.

318. *Final form of the disturbing function.*—Denote the coefficients of these functions by accented letters when the multiplication a'/r' has been made, so that θ is now an angle in the products of the series of § 292 by a'/r'. Let an angle in $\delta\rho'$, $\delta V'$ be ϕ, so that we have

$$\delta\rho'=\rho_c\cos\phi\,,\quad \delta V'=v_s\sin\phi\,,\quad m'=n'^2a'^3\,.$$

Putting α_1 for a/a' (§ 293), we obtain

$$R=\tfrac{1}{4}n'^2\mathrm{a}^2(-3)[(M_1'+\tfrac{3}{2}M_2')\rho_c\pm M_3'v_s+\alpha_1(M_6'+5M_8')\rho_c\pm\alpha_1(\tfrac{1}{4}M_7'+\tfrac{15}{4}M_9')v_s]\cos(\theta\pm\phi) \quad . \quad (4).$$

which is in the required form (§ 279).

For terms $\rho_s \sin\phi$ in $\delta\rho'$, $v_c \cos\phi$ in $\delta V'$, replace ρ_c by $\pm\rho_s$, $\pm v_s$ by v_c, $\cos(\theta\pm\phi)$ by $\sin(\theta\pm\phi)$.

The values of $\delta\rho'$, $\delta V'$ will be taken from NEWCOMB's tables of the Sun * (with some corrections). He tabulates $10^9 \log_{10}(1+\delta\rho') = 10^9 \log_{10} e \, . \, \delta\rho'$. If ρ_c, ρ_s denote his numbers, the parts in R which depend on these quantities must be multiplied by $10^{-9} \log_e 10$. The coefficients v_c, v_s are expressed in seconds of arc; I shall consider them as expressed in units of $0''\cdot 001$, so that the parts of R which depend on v_c, v_s must be multiplied by $10^{-3}/206265$. The formulæ of § 279, Chap. X., will then be available if we put $m''/m' = -3 \, . \, 10^{-9} \log_e 10 = -[\bar{9}\cdot 83934]$, so that

$$f = -[4\cdot 13292], \quad f' = -[\bar{1}\cdot 45682] \qquad (5),$$

and multiply v_c, v_s by the factor

$$10^{-3}/206265 \div 10^{-9} \log_e 10 = [\cdot 32335] \qquad (6).$$

Then A is the portion of (4) within square brackets, after v_s has been multiplied by this last factor.

Section (ii). *The Computation of $\delta\rho'$, $\delta V'$.*

319. *Forms of expression.*—In this chapter perturbations of the first order relative to the masses of the disturbing bodies are alone retained. If we had used the method of the variation of arbitrary constants to find $\delta\rho'$, $\delta V'$, the variations of the six elements of the solar orbit would have been obtained in the form

$$\alpha t + \beta + \Sigma k \cos(\lambda t + \mu),$$

where α, β, k, λ, μ are constants; in the coordinates we have a similar form, with the exception that in the elliptic terms k is of the form $k't + k''$ and $\alpha = 0$ in $\delta\rho'$. Further, we can put $\alpha = \beta = 0$ in $\delta V'$. All the periodic terms, except those which are independent of the argument of the disturbing planet, have therefore constant coefficients and are taken care of by the preceding method. Hence we have to consider only the terms

$$\left.\begin{aligned} \delta\rho' &= \delta a' + \Sigma(\rho_i + t\rho_i') \cos(il' + \alpha_i), \\ \delta V' &= \qquad\quad \Sigma(v_i + tv_i') \sin(il' + \alpha_i), \end{aligned}\right\} \quad (i = 1, 2, \ldots) \qquad (7).$$

320. *The non-periodic changes of the solar elements.*—Now the solar eccentricity is an observational quantity, and we can therefore choose our arbitraries such that $v_1 = 0$. The other v_i and ρ_2, ρ_3 are then so small that they may be neglected, and all the portions of the coefficients which depend on t may be expressed by a term $e'_1 t$ additional to e'. We have therefore only to add to the previous values of $\delta\rho'$ the term $\rho_1 \cos(l' + \alpha_1)$, which is treated in exactly the same way; $\delta a'$, which gives a constant term to R, and which is treated as in § 270 of Chap. X.; and, finally, the effect of a variation of e'. The mean motion of the solar perigee is not quite zero, and therefore dl'/dt is

* *Amer. Eph. Papers*, vol. vi., pt. I.

not quite equal to dT/dt; the only term sensibly affected in the Moon's motion is that with argument l', for which the divisor n' instead of $n' - d\varpi'/dt$ has been used.

The treatment of the variations of e', ϖ' require special methods; that of e' produces the well-known secular variations of w_1, w_2, w_3; these have been many times computed, and their theoretical values are not in doubt so far as the lunar equations are concerned. It also produces terms of the form $at+b$ in the coefficients of the periodic terms. These might be computed by means of the equations of variations, but I shall, in the next section, give another method which is much more simple for computation.

321. *Corrections to Newcomb's values.*—The values used in the solar tables (§ 318) are taken from his memoir * giving the computations. There are two sets of values in the memoir, obtained by independent computations, and the values of LEVERRIER are also given for comparison. These four sets of values were compared, and those in which the results agreed within the limits of accuracy required were accepted. But certain of the coefficients (1) in which NEWCOMB and LEVERRIER did not agree, (2) in which NEWCOMB's two sets of computations differed, and LEVERRIER's results were not given, (3) in which the degree of accuracy was not sufficiently high, or (4) in which the coefficients had not been obtained, have been recomputed. For this purpose the ordinary direct method was used—a method so well known † that it is unnecessary to do more than give the results; these are included in the tables of Sect. (v) below. Nearly every one of the few errors found was typographical and easily detected.

NEWCOMB has expressed doubts as to the sufficient accuracy of $\delta a'$ and the coefficients independent of the planetary arguments, and he has recomputed these portions. ‡ I have thought it worth while also to recompute these parts by a modification of the direct method, shown in the following section, which gives the required formulæ rapidly, instead of following the method of the variation of constants adopted by NEWCOMB.

322. *Computation of $\delta\rho'$ for the portion independent of the planetary arguments.*—We shall only need terms of the second order with respect to the planetary eccentricities and inclination in the constant term, and terms of the first order in the coefficient of the principal elliptic term. In order to get the former, we do not need the second elliptic term, since it can only produce a non-periodic term in combination with a term of the same argument and therefore one of the fourth order.

Dropping accents temporarily, we have for δr,

$$\frac{d^2}{dt^2}(r\delta r) + n^2\frac{a^3}{r^3}r\delta r = a\frac{dR}{da} + 2n\int\frac{dR}{dw_1}dt = (A + B\cos l + C\sin l)n^2a^2 \quad . \quad . \quad . \quad (8),$$

where R is the disturbing function of the Earth's motion due to a planet, and l is the Earth's mean anomaly. All the letters except t are supposed to be accented, and A, B, C are quantities whose squares may be neglected.

* *Amer. Eph. Papers*, vol. iii.

† See, *e.g.*, Cheyne's *Planetary Theory*, chap. vii.

‡ *Astron. Jour.*, No. 590.

Putting $a^3/r^3 = 1 + \frac{3}{2}e^2 + 3e\cos l$, we obtain by continued approximation for the particular integral corresponding to the terms on the right,

$$\frac{r\delta r}{a^2} = A(1 - \tfrac{3}{2}e^2) + \tfrac{3}{4}eCt + \tfrac{1}{2}t(B - 3Ae)\sin l - \tfrac{1}{2}tC\cos l \quad . \quad . \quad . \quad . \quad (9).$$

No arbitraries are necessary, since they will disappear in connection with corresponding arbitraries in δV.

323. The equation for the longitude is

$$na^2\sqrt{1-e^2}\,\delta V - 2\frac{d}{dt}(r\delta r) + \frac{dr}{dt}\delta r = -2a\int\frac{dR}{da}dt - 3n\int\!\!\int\frac{dR}{dw_1}dt^2 = (D\sin l + E\cos l + Ft)na^2 \quad . \quad (10),$$

suppose. Substituting for δr, we obtain, amongst others, terms of the form $\alpha t\cos l$, $\beta t\sin l$. These terms can be eliminated by supposing that ϖ, e receive increments $\delta\varpi$, δe proportional to the time; as we are not computing these increments they may now be neglected. The constant term only adds to the observed value of the mean longitude for $t=0$; it may therefore be dropped. Let δn, δe, δl_0 be the changes necessary in n, e, l for $t=0$, in order that the mean motion and the principal elliptic term may have same form as in undisturbed motion.

Then

$$(1 - \tfrac{1}{2}e^2)\delta V = t(F - \tfrac{1}{4}Be + \tfrac{3}{4}Ae^2 + \delta n - \tfrac{1}{2}e^2\delta n) + \cos l(E - C + 2e\delta l_0) + \sin l(D + B - 4Ae + 2\delta e)\,.$$

The coefficients of t, $\cos l$, $\sin l$, equated to zero, give δn, δe, δl_0.

Finally, substituting in

$$(1 + e^2 + 2e\cos l)\frac{r\delta r}{a^2} + \frac{\delta a}{a} - \delta e\cos l + e\delta l_0\sin l + \tfrac{1}{2}e\delta e\,,$$

which is the total addition to $\log r$ (that is, the required $\delta\rho'$), we find the terms

$$A(1 + e^2) + \tfrac{2}{3}F(1 + \tfrac{1}{2}e^2) - \tfrac{5}{12}Be - \tfrac{1}{4}De + \tfrac{1}{2}(B + D)\cos l + \tfrac{1}{2}(C - E)\sin l \quad . \quad . \quad . \quad (11).$$

Let

$$R = R_0 + R_c\cos l + R_s\sin l = R_0 + R_1\,,$$

and denote by I the operator ad/da. Then

$$n^2a^2(A + B\cos l + C\sin l) = a\frac{dR}{da} + 2\int\frac{dR}{dw_1}dt = IR_0 + (I + 2)R_1\,,$$

$$n^2a^2(F + D\cos l - E\sin l) = -2a\frac{dR}{da} - 3\int\frac{dR}{dw_1}dt = -2IR_0 - (2I + 3)R_1\,.$$

The expression (11) becomes on substitution of these values

$$n^2a^2[-\tfrac{1}{3}(1 - e^2)IR_0 + \tfrac{1}{12}e(I - 1)R_c - \tfrac{1}{2}(I + 1)R_c\cos l - \tfrac{1}{2}(I + 1)R_s\sin l]\,. \quad . \quad . \quad (12),$$

a simple form which it is easy to compute.

324. Let us now restore the accents and return to the usual notation. Then R

becomes the m''/Δ of § 294. The terms required may be taken directly from LEVERRIER'S expression (§ 298). We have, for outer planets,

$$R_0/\lambda a a'' = \tfrac{1}{2}A_0{}^0 + \tfrac{1}{4}(e'^2 + e''^2)(A_1{}^0 + A_2{}^0) - \tfrac{1}{2}\gamma''^2 B_0{}^1 - \tfrac{1}{2}e''e'(A_1{}^1 + A_2{}^1 - A_0{}^1)\cos(\varpi' - \varpi'')$$
$$= \tfrac{1}{2}A_0{}^0 + \tfrac{1}{8}B_0{}^1(e'^2 + e''^2) - \tfrac{1}{2}\gamma''^2 B_0{}^1 - \tfrac{1}{4}B_0{}^2 e''e' \cos(\varpi' - \varpi''),$$
$$R_c/\lambda a a'' = -\tfrac{1}{2}e'A_1{}^0 + \tfrac{1}{2}e''(A_1{}^1 - A_0{}^1)\cos(\varpi' - \varpi''), \quad R_s/\lambda a a'' = -\tfrac{1}{2}e''(A_1{}^1 - A_0{}^1)\sin(\varpi' - \varpi''),$$

where I use the notation of Sect. (ii), Chap. XI., and certain relations* to reduce the expression for R_0; also $\lambda = m''/m'$. The required formula for $\delta\rho'$ becomes, on making use of the relation $I\phi_p = (p+1)\phi_{p+1} + p\phi_p$, satisfied by $\phi = A$, $\phi = B$, for outer planets,†

$$\delta\rho' = \tfrac{1}{3}\frac{m''}{m'}aa''\left\{-\tfrac{1}{2}A_1{}^0 + \frac{e'^2}{4}(2A_1{}^0 - A_2{}^0) - \frac{e'^2 + e''^2}{B}B_1{}^1 + \tfrac{1}{2}\gamma''^2 B_1{}^1 + \frac{e'e''}{4}(B_1{}^2 + A_2{}^1 - \tfrac{1}{2}A_1{}^1 + \tfrac{1}{2}A_0{}^1)\cos(\varpi' - \varpi'')\right.$$
$$+ \frac{m''}{m'}aa''\left\{\frac{e'}{2}(A_2{}^0 + A_1{}^0) - \frac{e''}{2}(A_2{}^1 + \tfrac{1}{2}A_1{}^1 - \tfrac{1}{2}A_0{}^1)\cos(\varpi' - \varpi'')\right\}\cos l'$$
$$+ \frac{m''}{m'}aa''\left\{\qquad\qquad \frac{e''}{2}(A_2{}^1 + \tfrac{1}{2}A_1{}^1 - \tfrac{1}{2}A_0{}^1)\sin(\varpi' - \varpi'')\right\}\sin l'.$$

For inner planets, R_0 only requires the change a' for aa'', but R_c, R_s are given by

$$R_c/\lambda a' = \frac{e'}{2}(A_1{}^0 + A_0{}^0) - \frac{e''}{2}(2A_0{}^1 + A_1{}^1)\cos(\varpi' - \varpi''), \quad R_s/\lambda a' = \frac{e''}{2}(2A_0{}^1 + A_1{}^1)\sin(\varpi' - \varpi''),$$

while in the expression (12) for $\delta\rho'$ we put $-I-1$ for I.

The values of $a''A_p^i$, $a''B_p^i$, $a'A_p^i$, $a'B_p^i$ are given in the auxiliary tables for the direct action.‡ Those of the other quantities are found in Sect. (iv), Chap. XI. The final results are included with the other terms in $\delta\rho'$, $\delta V'$.

Section (iii). *Second Method. Application to Non-periodic Changes.*

325. *Statement and Solution.* §—The method may be regarded as a particular case of the general problem of four bodies, or of three bodies, or as a general method for treating any motion which is transmitted through one body to another, according to the view we wish to adopt. The last view will be that most convenient for our immediate purposes.

Suppose that we have been able to solve, in terms of t and arbitrary constants, a dynamical problem which has a force function F. This function, expressed initially in terms of the coordinates, may also contain t explicitly and given constants. I shall suppose that it contains t explicitly only through certain functions of the given constants, u_h, some of which may therefore be constant and some variable. Now suppose that, owing to some external agency, the u_h are not the complete values of these

* *A.P.E.*, sect. iii.

† The formulæ do not quite agree with those of NEWCOMB given in *Astr. Jour.*, No. 590, but the numerical results agree with his as given in his paper, *Carn. Inst. Publ.*, No. 72, p. 90.

‡ *A.P.E.*, sect. v.

§ I have given the method in a paper in the *Trans. Amer. Math. Soc.*, vol. vi. pp. 332–343.

given functions, but require certain additions, δu_h, whose values in terms of the time are given. The ordinary method of treatment consists in substituting these new values in F and obtaining a disturbing function * $\Sigma_h(\delta u_h dF/du_h)$; this is accounted for by finding what variable values must be given to the arbitraries, so that when these values are substituted instead of the constant values in the expressions for the coordinates and velocities, we shall have the complete solution of the problem. In this method no account is taken of the fact that F retains the same *form* with respect to the u_h whatever *values* may be given to these functions.

In the memoir referred to, an attempt was made to utilise the absence of change in the form of F by considering the problem in the following way:—To find the variations of the arbitrary constants when not only *their* variations, but *also those of the* u_h, are substituted in the expressions for the coordinates and velocities. I proved that if this plan were followed, the solution was equivalent to adding a disturbing function

$$R = \Sigma_h U_h\left(\frac{d}{dt}\delta u_h - \delta\frac{du_h}{dt}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where the U_h are defined by the differential equations

$$\frac{dU_h}{dt} = -\frac{\partial F}{\partial u_h} - \Sigma_k U_k \frac{\partial}{\partial u_h}\frac{du_k}{dt} - \frac{\partial B}{\partial u_h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

it being supposed that F is expressed in terms of the coordinates and the u_h, that du_h/dt is expressible in terms of the u_h, and that B is expressed as a function of the constants and of those u_h which are independent of t.

326. *Application to the secular changes of* e', ϖ'.—We have initially e', ϖ' constant. Let $\delta e' = e_1'e't$, $\delta\varpi' = \varpi_1't$. The u_h are n', $n't+\epsilon'$, ϖ', e', and therefore ϖ', e' are independent of the other u_h and of du_h/dt. Also (*loc. cit.*) $U_{e'}$, $U_{\varpi'}$ contain no non-periodic terms. Hence

$$R = -e_1'\int e'\frac{\partial F}{\partial e'}dt - \varpi_1'\int\frac{\partial F}{\partial\varpi'}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

in which the non-periodic term arising from $\partial F/\partial e'$ must be dropped.

If we substitute this value of R in the equations of Chap. X., *as we have no non-periodic part of* R, $\delta c_i = 0$, $\delta w_i = 0$, and therefore the secular accelerations are obtained by putting $e'(1+e_1't)$ for e' in the values of c_1, c_2, c_3 expressed as functions of n, e, γ, e', n', and finding the values of n, e, γ which result.† The motion of ϖ' produces nothing in this connection, since it is not present in the c_i.

I have shown ‡ that if we neglect α_1^2, a quantity which is quite insensible, the variations of n, e, γ can also be obtained from the equations

$$c_2\delta\frac{db_2}{d\alpha_i} + c_3\delta\frac{db_3}{d\alpha_i} = \tfrac{3}{2}\delta\frac{d}{d\alpha_i}\left(\frac{\mu}{r}\right)_0, \quad \alpha_i = n, e, \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

* For simplicity, only variations of the first order are retained, but the methods are applicable when we take in higher powers.

† This is Newcomb's theorem, *Amer. Eph. Papers*, vol. v., pt. 3, p. 191.

‡ *Proc. Lond. Math. Soc.*, vol. xxviii. p. 154.

where $(\mu/r)_0$ denotes the non-periodic of μ/r, the functions being expressed in terms of n, c, γ, e', n', the first four only receiving variations, that of e' being given.

327. There remain the periodic terms of R. Since e', ϖ' occur in R only through r', V', we have only to put the periodic terms

$$\delta\rho' = -e_1' \cdot e' \frac{d\rho'}{de'} - \varpi_1' \frac{d\rho'}{d\varpi'}, \quad \delta V' = -e_1' e' \frac{dV'}{de'} - \varpi_1' \frac{dV'}{d\varpi'}, \qquad . \quad . \quad . \qquad (17)$$

in the formulæ of § 317, and, after rejecting all non-periodic terms, integrate; the resulting disturbing function consists only of periodic terms.

The variations of the elements are then substituted in the coordinates. In accordance with the principle of the method, we must also put $e'(1+e_1't)$, $\varpi' + \varpi_1't$ for e', ϖ', in order to obtain the true values of the coordinates.

No other secular terms can be produced from the secular variations of the solar elements. It will be shown in Sect. (iv) that those of the inclination and node only produce periodic variations.

The method of this section might have been used for all the indirect perturbations.

Section (iv). *The Motion of the Ecliptic.*

328. *Choice of the Mean Ecliptic.*—Owing to the action of the planets on the Earth, the plane of the Earth's orbit is not fixed, but has a motion which can be expressed as secular and periodic variations of the inclination and longitude of the node with reference to some fixed plane. I choose as fixed plane the ecliptic at the date 1850·0, and refer the motion of the Moon to the *mean ecliptic* at time t.* The periodic perturbations are then included in the terms of the Sun's disturbing function which depend on z', portions which have been previously neglected (Chap. I., § 3). To be included in this mean ecliptic are one or two minute inequalities of very long period which then give rise to no terms in the Moon's motion, but which would do so if included in z'.

329. *The disturbing function for the moving ecliptic.*—This is most easily found in a general manner. Let x, y, z, u, v, w be the coordinates and velocities of a particle of mass m, referred to rectangular axes which have velocities θ_1, θ_2, θ_3 about themselves, and let the force function be denoted by mF. The equations of motion of m are then given by

$$\frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z},$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial u}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w},$$

where

$$H = \tfrac{1}{2}(u^2 + v^2 + w^2) - F - R,$$

$$R = vx\theta_3 - wx\theta_2 + wy\theta_1 - uy\theta_3 + uz\theta_2 - vz\theta_1,$$

if we assume that θ_1, θ_2, θ_3 are independent of x, y, z, u, v, w.

* I have discussed this point in vol. lxviii. pp. 450–455, of the *Monthly Notices*, and have also given there the substance of this section.

When $R=0$, the equations become the same as those referred to fixed axes, and therefore R is the disturbing function for the motion of the axes.

Let i' be the inclination of the moving ecliptic (xy plane) to that of 1850·0, τ the longitude of the node on the fixed ecliptic, and for the origin of longitudes on the moving ecliptic take the "departure point" whose distance from the node on the moving ecliptic is the same as the distance from the node of the origin of reckoning on the fixed ecliptic. Then by Euler's equations

$$\theta_1=\frac{di'}{dt}\cos\tau-\frac{d\tau}{dt}\sin i'\sin\tau\,,\quad \theta_2=\frac{di'}{dt}\sin\tau+\frac{d\tau}{dt}\sin i'\cos\tau\,,\quad \theta_3=\frac{d\tau}{dt}(\cos i'-1) \quad . \quad (18).$$

The values of i', di'/dt, $d\tau/dt$ are small quantities of the first order, so that their squares and products may be neglected. On substituting the values, so limited, in R, we find the factor di'/dt common to all the terms, and therefore, since u, v, w differ from dx/dt, dy/dt, dz/dt by quantities of the same order,

$$R=\frac{di'}{dt}Q\,,\quad Q=\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)\cos\tau-\left(x\frac{dz}{dt}-z\frac{dx}{dt}\right)\sin\tau \quad . \quad . \quad . \quad . \quad (19).$$

This disturbing function is available for *any* moving ecliptic so long as we may neglect the squares of its perturbations, but under the assumptions of § 328 we substitute for i' only its secular part. The resulting disturbing function I denote by R_1.

330. *The disturbing function for perturbations of the Earth out of the plane of reference* is, if we neglect squares of z' and the terms dependent on a,

$$R=\frac{m'}{r'^3}\cdot\frac{3(xx'+yy')zz'}{r'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

by § 3, Chap. I. With the notation and limitations of the previous section,

$$z'=i'(y'\cos\tau-x'\sin\tau) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Since this expression has the small factor i', we consider τ as a constant. In the paper referred to in § 328, I have shown that the expression (20) for R can be transformed into

$$R=i'\left\{\left(z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}\right)\cos\tau-\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)\sin\tau\right\}=-i'\frac{dQ}{dt} \quad . \quad . \quad . \quad (22),$$

which again is a perfectly general expression for the disturbing function when we can neglect squares of the perturbations. The value of Q is that given in the previous paragraph, and therefore the computation of one function serves for both disturbances. Under the assumptions of § 328 we substitute for i' in (22) only its periodic part, neglecting the minute perturbations of i' which are of long period relatively to that of the Moon's node. The disturbing function thus limited will be denoted by R_2.

331. *Computation of Q.*—We have

$$Q=\frac{dz}{dt}\rho\sin(V-\tau)-z\frac{d}{dt}\{\rho\sin(V-\tau)\}$$

$$=\text{real part of } (n-n')[Dz\cdot u\zeta^{-1}e^{w_1\iota}-zD(u\zeta^{-1}e^{w_1\iota})]e^{-\tau\iota}$$

$$=\qquad ,, \qquad \frac{na^2}{\iota}k\left[\frac{2u\zeta^{-1}\cdot D(\iota z)-(D+1+m)(\iota z\cdot u\zeta^{-1})}{k(1+m)a^2}\right]\epsilon^{(w_1-\tau)\iota}.$$

The expansion of the portion in square brackets has been given in § 291, Chap. X. Let M exp. $\theta_1\iota$ be one of the terms of this expansion. Then one of the terms of Q is given by

$$Q=na^2kM\sin(\theta_1+w_1-\tau)=na^2kM\cos\left(\theta_1+w_1-\tau-\frac{\pi}{2}\right).$$

332. *Computation of R_1.*—Let $di'/dt=p$. Then

$$R_1=\frac{4p}{n'}\cdot\frac{k}{m}\left[\tfrac{1}{4}n'^2a^2M\cos\left(\theta_1+w_1-\tau-\frac{\pi}{2}\right)\right],$$

and therefore if we put m''/m' for the factor outside the square brackets the equations of § 279, Chap. X., can be immediately applied; here $A=M$.

The value of p is* $0''\cdot4710$ per annum, $n'=3548''\cdot2$ per diem, and therefore

$$\frac{m''}{m'}=\frac{4(\cdot4710)k}{(365\cdot25)(3548\cdot2)m}=[\bar{7}\cdot9397],$$

giving

$$f=[6\cdot2333],\qquad f'=[1\cdot5572].$$

In finding derivatives with respect to n, we must use $d(nka^2A)/dn$.

333. *Computation of R_2.*—Let $P\cos\phi$ be any term of i' reckoned in units of $0''\cdot001$, and s_1 the number of seconds in the daily motion of the angle $\theta_1+w_1-\tau=\theta$. Then, using the first form for Q, we have

$$R_2=\frac{m''}{m'}\tfrac{1}{4}n'^2a^2A\cos(\theta\pm\phi),$$

where

$$A=-\tfrac{1}{2}s_1MP,\quad \frac{m''}{m'}=\frac{4}{1000}\,\frac{1}{206265}\,\frac{1}{3548\cdot2}\,\frac{k}{m}=+[\overline{12}\cdot5149].$$

The computation of derivatives with respect to n must be made with s_1Ma^2.

The last fact makes the computation easier with formulas (20), (21). We have

$$R_2=\tfrac{3}{2}n'^2i'\left(\frac{a'}{r'}\right)^3\rho z\{\sin(V-\tau)-\sin(V-2V'+\tau)\}$$

$$=\tfrac{1}{4}n'^2a^2\frac{m''}{m'}\{-M_4'P\cos(\theta\pm\phi)\}\text{ with } h''=\tau,$$

$$+\tfrac{1}{4}n'^2a^2\frac{m''}{m'}\{M_4''P\cos(\theta\pm\phi)\}\text{ with } h''=2V'-\tau,$$

using the third formula of § 285, Chap. X., after multiplication by a'/r'. For these

* Leverrier, *Ann. Obs. Paris*, vol. iv. p. 50, after correction for the adopted masses of the planets.

terms it is sufficiently accurate to neglect the solar eccentricity in finding the Moon factors; this gives $M'_4 = M''_4 = M_4$, $V' = \mathrm{T}$. The value of m''/m' is

$$\frac{m''}{m'} = \frac{3}{1000} \cdot \frac{1}{206265} = [\bar{8}{\cdot}1627].$$

The terms, arising from these disturbing functions, in the results at the end of this chapter, are those in longitude which contain the argument w_3 *explicitly*, and those in latitude which contain the argument w_1 *explicitly*. The most important are two of the latter with arguments $w_1 \pm (5\mathrm{T} - 3\mathrm{V})$ and having coefficients $0''{\cdot}077$ and $0''{\cdot}030$ respectively.

Section (v). *Numerical Values of the Earth's Perturbations.*

334. *Sources.*—The general values for $\delta\rho'$, $\delta V'$ are taken from NEWCOMB'S tables of the Sun (§ 318), with some corrections and additions (§ 321); the secular variations of e', ϖ' are from the same source. The terms independent of the planet's arguments are found in § 324. The values of i', τ are from LEVERRIER,* after correction for the masses of the planets adopted here; the coefficient of $2\mathrm{J} - 2\tau$ in i' was recomputed.

335. *Notation.*—The values of $\delta\rho'$, $\delta V'$ are given by NEWCOMB as cosines and sines of the mean anomalies; it was convenient to retain them in this form. LEVERRIER gives the values of i' in terms of literal arguments; these were combined and expressed in terms of the mean longitudes, which is the final form for all the inequalities due to planetary action (Chap. XV.). Hence, I put

$$\begin{matrix}\delta\rho' \\ \delta V'\end{matrix} = \begin{matrix}\rho_c \\ v_c\end{matrix} \cos\{j'(\mathrm{T} - \varpi') + j''(\mathrm{T}'' - \varpi'')\} + \begin{matrix}\rho_s \\ v_s\end{matrix} \sin\{j'(\mathrm{T} - \varpi') + j''(\mathrm{T}'' - \varpi'')\}, \quad i' = i_c \cos(j'\mathrm{T} + j''\mathrm{T}'' + \alpha^\circ).$$

Also, v_c, v_s are expressed in units of $0''{\cdot}001$, while ρ_c, ρ_s are expressed in absolute units multiplied by $10^9 \log_{10} e$, that is, they are the coefficients in $10^9 \log_{10}(1 + \delta\rho)$.

For the secular terms, $de'/dt = e_1'e'$, $d\varpi'/dt = \varpi_1'$.

* *Ann. Obs. Paris*, vol. iv. p. 50.

336. *Venus*, T″ = V.

j'	j''	ρ_c	ρ_s	v_s	v_c	i_c	α
0	0	+ 627					
1		+ 14	− 7				
2		...	...	...	...	24	209
0	− 1	− 85	− 39	− 67	+ 33		
1		− 2062	− 1146	− 4228	+ 2353	24	180
2		+ 68	− 14	− 34	− 65		
3		+ 14	− 8	− 8	− 3	30	209
0	− 2	0	+ 4	+ 1	− 3		
1		+ 84	+ 136	+ 60	− 99		
2		+ 3593	+ 5822	+ 2903	− 4702		
3		− 596	− 632	− 1737	+ 1795		
4		+ 40	+ 33	− 33	+ 30	43	209
2	− 3	0	+ 21	+ 1	− 13		
3		+ 44	+ 1044	+ 27	− 666		
4		− 381	− 1448	− 397	+ 1508		
5		+ 126	+ 148	− 684	+ 763	201	209·4
6		+ 14	+ 13	− 12	+ 12		
3	− 4	0	+ 6	− 1	− 3		
4		− 166	+ 337	− 93	− 188		
5		− 51	+ 189	− 38	− 139		
6		− 25	− 91	− 42	+ 146	39	29
7		+ 3	+ 5	− 4	+ 5		
5	− 5	− 134	+ 93	− 69	− 47		
6		− 39	+ 43	− 25	− 28		
7		− 37	+ 136	− 33	− 119		
8		+ 0·3	− 27·9	− 1	+ 154	20	299
6	− 6	− 80	+ 8	− 38	− 4		
7		− 24	+ 7	− 13	− 4		
8		− 10	+ 10	− 7	− 6		
9		+ 3	− 12	+ 3	+ 14		
7	− 7	− 38	− 17	− 18	+ 8		
8	− 8	− 14	− 19	− 7	+ 9		
12		− 43·2	+ 8·1	− 41	− 8		
13		+ 9·00	− 7·71	+ 1416	+ 1251		
14		− 25·2	+ 22·3	+ 24	+ 21		
16	− 18	*see* § 350.					

337. *Mars*, T″ = M.

j'	j''	ρ_c	ρ_s	v_s	v_c	i_c	a
0	0	− 13					
1		+ 5	− 3				
− 2	1	− 5	+ 6	− 4	− 5		
− 1		− 92	+ 119	− 167	− 216		
0		+ 27	− 6	− 47	− 8		
− 3	2	− 13	− 50	− 10	+ 40		
− 2		− 573	− 1976	− 567	+ 1963		
− 1		+ 64	− 137	− 617	− 1659		
0		− 18	− 25	+ 15	− 24		
− 3	3	− 154	− 67	− 118	+ 53		
− 2		− 77	− 201	− 153	+ 396		
− 4	4	+ 46	− 17	+ 32	+ 11		
− 3		+ 461	+ 125	+ 483	− 131		
− 2		+ 43	+ 96	− 256	+ 526	6	263
− 1		+ 6	+ 8	− 5	+ 7		
− 4	5	+ 87	− 62	+ 69	+ 49		
− 3		+ 87	+ 17	+ 200	− 38		
− 5	6	− 3	+ 30	− 2	− 20		
− 4		− 102	+ 94	− 113	− 104		
− 3		− 27	− 4	+ 100	− 11		
− 5	7	+ 4	+ 60	+ 3	− 49		
− 4		− 26	+ 28	− 72	− 78		
− 6	8	− 12	− 9	− 8	+ 6		
− 5		− 8	− 44	− 10	+ 51		
− 4		+ 5	− 6	− 12	− 17		
− 6	9	− 30	− 16	− 25	+ 13		
− 5		− 4	− 17	− 15	+ 60		
− 7	10	+ 7	− 3	+ 5	+ 2		
− 6		+ 14	+ 6	+ 18	− 7		
− 7	11	+ 17	− 10	+ 15	+ 9		
− 6		+ 8	+ 3	+ 42	− 12		
− 8	13	− 1	+ 15	− 1	− 13		
− 7		− 4	+ 3	− 33	− 30		
− 9	15	− 17	− 14	− 16	+ 13		
− 8		− 0·90	− 5·92	− 3*	+ 20*		
− 9	17	− 1·3	− 0·6	+ 24	− 10		

* The last figure was not computed, and is not needed.

338. *Jupiter*, T″ = J.

j	j''	ρ_c	ρ_s	v_s	v_c	i_c	a
0	0	− 513					
1		+ 39	+ 22				
− 3	1	− 2	+ 5	− 1	− 3		
− 2		− 78	+ 193	− 52	− 155		
− 1		+ 56	+ 7067	+ 59	− 7208		
0		+ 227	− 89	− 2582	− 307	6	354
1		+ 79	+ 9	− 73	+ 8		
− 3	2	+ 102	− 17	+ 68	+ 11		
− 2		+ 4021	− 203	+ 2728	+ 136	13·3	180
− 1		+ 1376	+ 486	+ 1518	− 537		
0		− 1	− 8	− 70	− 22	152	162
− 3	3	+ 43	+ 278	+ 27	− 162		
− 2		+ 796	− 104	+ 551	+ 71		
− 1		+ 172	+ 26	+ 208	− 31		
0		...	...	...	...	18	150
− 4	4	− 29	+ 5	− 16	− 3		
− 3		+ 13	+ 73	+ 9	− 43		
− 2		+ 110	− 24	+ 78	+ 17		
− 1		+ 17	+ 1	+ 23	− 1		

339. *Saturn*, T″ = S.

j	j''	ρ_c	ρ_s	v_s	v_c
0	0	− 24			
− 2	1	+ 15	+ 3	+ 11	− 3
− 1		+ 422	+ 79	+ 412	− 77
0		+ 7·89	− 0·53	− 320	− 3
1		+ 8	0	− 8	0
− 2	2	− 152	− 57	− 101	+ 38
− 1		− 103	− 44	− 103	+ 45
0		+ 0·31	− 0·56	− 17	+ 2
− 2	3	− 30	− 11	− 20	+ 7
− 1		− 16	− 6	− 16	+ 6

340. *Mercury*, T″ = Q.

j	j''	ρ_c	ρ_s	v_s	v_c
0	0	− 27			
1		− 5	+ 2		
4	− 1	− 2	− 4	− 13	+ 19

341. *All planets. Secular terms.*

$\delta w_1' = 0$

$\varpi_1' = +[\bar{6}\cdot 9537]n'$

$\delta n' = 0$

$e_1' = -[\bar{6}\cdot 5968]n'$

$\tau = 173^\circ\cdot 46$

$i' = +[\bar{7}\cdot 5604]n't$

Section (vi). *A Sieve for the Rejection of Insensible Coefficients.*

342. *Terms to be considered.*—Just as with the direct inequalities (Sect. (iii), Chap. XI.), we only need to consider the possible terms of long period. Also, as before, I consider separately those that do or do not contain w_1 in their arguments. The latter can be at once dealt with; the number of possible terms in the lunar factors is practically limited to those with arguments $2D-2l$, $2D-2F$, $2F-2l$, $D-l$, and in any case it is a brief matter to consider the possible combinations with the planetary arguments of all but $2D-2l$, $2D-2F$, which are computed in the regular course with the short-period terms.

Those that contain w_1 in their arguments can only produce sensible terms in combination with planetary factors having nearly the same period, that is, a month. Hence we have only to consider the values of coefficients in $\delta\rho'$, $\delta V'$ belonging to terms which have periods of about a month or less.

343. *Construction of the Sieve for terms containing w_1.*—The equation for $\delta r'$ is (§ 322)

$$\frac{d^2}{dt^2}(r'\delta r') + \frac{n'^2a'^3}{r'^3}r'\delta r' = a'\frac{dR'}{da'} + 2n'\int\frac{dR'}{dT}dt \quad . \quad . \quad . \quad . \quad . \quad (23),$$

where $R' = m''/\Delta$; the other portion of R' gives coefficients which are quite insensible in the class of terms considered here. Let q be the mean motion of a short-period argument in R', and therefore in $\delta r'$; for the periods approximating to a month in $\delta r'$ we have $n'/q = m$ approximately.

As far as the effect of the periods is concerned, the four terms of (23) are of relative orders q^2, n'^2, n'^2, $2n'q$. And further, the largest terms in R' with periods approximating to a month must have high multiples of $T-T''$ in their arguments, and in this case the two derivatives of R' are of the same order of magnitude. Hence the order of $\delta\rho'$ for such terms is given by

$$n'^2a'^2\delta\rho' = -a'\frac{dR'}{da'}\frac{n'^2}{q^2} \quad \text{or} \quad -a'\frac{dR'}{da'}m^2 \quad . \quad . \quad . \quad . \quad . \quad (24).$$

The equation for $\delta V'$ may be written

$$n'a'^2\sqrt{1-e'^2}\frac{d}{dt}(\delta V') = -2\frac{n'^2a'^3}{r'^3}(r'\delta r') - \frac{d}{dt}\left(\delta r'\frac{dr'}{dt}\right) + n'\int\frac{dR'}{dT}dt \quad . \quad . \quad . \quad (25).$$

Similar reasoning shows that the order of the right-hand member is the order of its last term, so that, on integration, $\delta V'$ is seen to be of the same order of magnitude as $\delta\rho'$.

Now $\Delta^2 = r'^2 + r''^2 - 2r'r''\sigma$, where σ is the cosine of the angle between r', r''. Hence

$$a'\frac{dR'}{da'}r' = \frac{dR'}{dr'} = m''\left(-\frac{r'^2-r''^2}{2\Delta^3} - \frac{1}{2\Delta}\right) = \text{order of } \frac{m''}{3}\cdot\frac{a'^2}{\Delta^3} \quad . \quad . \quad . \quad (26),$$

for the worst case, that of Venus. Hence $3\delta\rho'$ is of order $(m''/m')(a'^3/\Delta^3)m^2$.

Take first the inequalities depending mainly on $\delta\rho'$. A comparison of the disturbing functions in §§ 294, 317 shows that the order for the indirect terms is to that for the direct terms as $3\delta\rho' : (m''/m')(a'^3/\Delta^3)$, that is, $m^2 : 1$, since we may take $3\zeta^2/\Delta^5$ as having the same order at the worst as $1/\Delta^3$. Hence any term which is shown by calculation to have a coefficient less than $1''$ due to direct action will not be sensible in the indirect action. There is only one term left, that with argument $l+16\mathrm{T}-18\mathrm{V}$, coefficient $14''\cdot 55$; the order of the coefficient for the indirect action is this number multiplied by m^2, that is, the order $0''\cdot 09$ (the computed coefficient found below is $0''\cdot 06$).

The terms due to $\delta V'$ are treated in exactly the same way and give similar results. The direct inequalities are all small, and there are no sensible ones arising from the indirect action.

Hence, *there are no sensible terms of long period containing the argument w_1 and arising from indirect action in the plane of reference, except a small term having the argument of the great Venus inequality.*

344. *Terms arising from i'.*—The principal argument in the Moon factors is w_3, and the combinations of this with the comparatively few terms in i' which are sensible are first studied; then the terms of one order higher with respect to the lunar eccentricity and inclination, and so on. It soon becomes quite obvious that the only terms beyond those of lowest order in the Moon factors must be of long period relatively to that of the Moon's node. As the terms in $1/\Delta$ to be considered must have the factor γ''^2, their number is very limited.

The methodical search for long-period terms was simple. The tables formed for the periods in the sieve for the direct terms were available.* As w_3 is itself of long period, it was only necessary to combine w_3 with terms in that table which contained multiples of w_2-w_3, w_3 which were either both even or both odd, that is, terms for which the multiples of w_3 or F were even. The combinations of these with w_3 were those required. Only two survived, apart from those with the sole lunar argument w_3, namely, the arguments

$$2w_2+w_3-2\mathrm{J}=2\mathrm{D}-2l-w_3-2\mathrm{J}+2\mathrm{T}, \quad \text{period } 277 \text{ years;}$$
$$w_2-w_3-2\mathrm{J}=2\mathrm{F}-\mathrm{D}-l+w_3-2\mathrm{J}-\mathrm{T}, \quad \text{period } 540 \text{ years.}$$

The latter is quite small compared with the former, on account of the lunar characteristic $\mathrm{k}^3\alpha_1\mathrm{e}$ as compared with $\mathrm{e}^2\mathrm{k}$, and the planetary factor is also much smaller; the period is twice as long, but this only multiplies the relative coefficient by 4.

In a note at the end of a paper lately published,† I gave a value for the coefficient of the former term as $0''\cdot 21$. Since the paper was published the term has been recomputed with the disturbing function (20); this revealed an error in the former computation, and the coefficient appears to be $0''\cdot 003$. It is therefore not retained in the final results.

* *A.P.E.*, sect. v.

† *Monthly Notices*, vol. lxviii. p. 170.

Section (vii). *Computation of the Lunar Perturbations.*

345. *The disturbing functions* are given in §§ 318, 324, 327, 332, 333, the values of the planetary factors in the last section, and those of the lunar factors in Sect. (v) of Chap. X. The method of arrangement was to take one lunar argument with all multiples of $T - \varpi'$ and form the products for all the planetary arguments. The equations of variations have been so arranged that the process of finding the values of δw_i, δn, δc_2, δc_3 from the disturbing functions is very brief and simple, inasmuch as it was rarely necessary to compute more than two of these six variations. In fact, in the few cases where more than two were needed, a simple ratio, the same for all terms, generally sufficed; such a ratio was also sufficient in the majority of cases to find all the other variations after the principal one had been obtained.

The experience gained in computing the direct inequalities suggested that the work could be much abbreviated by considering the peculiarities of each lunar argument, and these peculiarities are set forth in the following paragraphs.

346. *The primaries independent of the lunar angles.*—Here $\delta n = \delta c_2 = \delta c_3 = 0$, and δw_1 is first computed, and then $-e\delta w_2$, so that the secondary arising from the substitution of δw_2 in the principal elliptic term, $-2e \cos l \,.\, \delta w_2$, is obtained directly. For δw_3, it was sufficiently accurate to treat $\delta w_3/\delta w_2$ as a constant which is the same for all terms, and indeed for the small terms, $\delta w_2 : \delta w_3 : \delta w_1$ are constant ratios with sufficient accuracy.

When these primaries and secondaries have been found, the remaining secondaries can be written down almost by inspection. I therefore only give the coefficients of the primaries arguments ψ, and of the secondaries arguments $\psi \pm l$.

The primaries of very long period arising from terms of very long period in $\delta\rho'$, $\delta V'$ are treated in § 352 below.

347. *Primaries containing w_1 and independent of w_2, w_3.*—Here the periods of all the terms are very nearly the same, δw_2 is nearly equal to δw_1, and δw_3 is about $\frac{1}{6}\delta w_2$. Hence the secondaries are all very small, the largest being less than $0''{\cdot}010$. All of them greater than $0''{\cdot}001$ were computed, and will be included in the final results, but it is unnecessary to print in detail any but the primaries. The principal variation is again δw_1.

348. *Primaries containing w_1, w_2 or w_2 only.*—These are the terms in which the secondaries are generally much more important than the primaries, and in which the theorem of § 282, Chap. X., has its full force, the principal variations being δw_2, δc_2. If the primary contain w_2 in the form $il + \phi$ (i positive), the principal secondary is that with argument $(i-1)l + \phi$, and this was first computed in all cases; then the value of δw_1; δw_3 was insensible, or only produced very small terms. The terms divide into two classes according as their periods approximate to a month and less, or are much longer.

The first class includes the lunar arguments

$$l, \quad 2D - l, \quad 2D + l, \quad 4D - l, \quad 2l,$$

for which the principal terms, those in longitude, have the lunar arguments

$$0, \quad 2D, \quad 2D, \quad 4D, \quad l,$$

respectively.

The second class contains only one argument, $2D - 2l$, for which the principal term in longitude has the argument $2D - l$.

I give δw_1, that is, the primary, and the coefficients of the principal set of terms in longitude only; from these all others can be obtained immediately.

349. *Primaries containing w_1, w_3 or w_3 only.*—The principal variations being δw_3, δc_3, the largest term is always the principal secondary in latitude, and the statements of the previous section can be repeated. But only two lunar arguments have to be considered: $2F$, which belongs to the first class, and $2F - 2D$, which belongs to the second class. The principal terms in latitude contain the lunar arguments F, $F - 2D$, and the coefficients of these four terms are alone set down.

350. *Special terms.*—Beyond the inequalities mentioned above, there are two with arguments

$$D - l - 4T + 3V, \quad l + 16T - 18V,$$

and with periods of 94 years and 273 years. The former is computed in the same manner as the other terms. For the latter we require to find $\delta\rho'$, $\delta V'$ for the argument $16T - 18V$ or $16(T - \varpi') - 18(V - \varpi'')$. It was known (§ 343) that the final coefficient was of the order of $0''\cdot 1$. It was therefore sufficient to find the terms with multiples 16, 17, 18 of $T - \varpi'$ combined with $18(V - \varpi'')$, so that they might be combined with the lunar arguments l, $l - (T - \varpi')$, $l - 2(T - \varpi')$ in order to give the required argument. The direct method of computation gave (in absolute units), with sufficient accuracy,

$$\begin{matrix} \delta\rho' \\ \delta V' \end{matrix} = \begin{matrix} -\cdot 0000059 \cos \\ -\cdot 0000046 \sin \end{matrix} \left\{ 16(T - \varpi') - 18(V - \varpi'') + 129^\circ \right\} \cdot \frac{m''}{m'} \qquad (27).$$

Thence arises the only sensible term,

$$\delta w_1 = -0''\cdot 06 \sin (l + 16T - 18V + 150^\circ).$$

351. *Omitted terms.*—The portions of the disturbing function independent of the planet's mean motion are the constant term and those with argument l'. The former chiefly produce small changes in the mean motions of the perigee and node, and also slight changes in n, e, γ which affect the coefficients of the evection and variation, but the latter are quite insensible. The variation of e' produces the secular accelerations.

The terms due to the motion of the ecliptic and the latitude of the Sun do not combine with any others due to $\delta\rho'$, $\delta V'$.

The detailed results given in the next section do not contain these terms, nor the terms in the two following paragraphs: they will all be given in the final results due to indirect action contained in the last section of this chapter.

352. *Terms left as perturbations of the elements.*—A few of the primaries which have very long periods and which are independent of the lunar angles are

so left, and their values are to be found with the collected results at the end of this chapter. The principal inequality dne to the motion of the mean ecliptic is also treated in this way.

But there are short-period primaries due to long-period terms in $\delta\rho'$, $\delta V'$ which are sensible. I shall show that these may be accounted for by *the substitution of* w_1' + long-period terms in $\delta V'$ *for* w_1' *in the final expressions for the coordinates of the Moon.*

Consider how such terms would be treated by the second method in Sect. (iii). The chief perturbations of the solar coordinates arise from $\delta n'$ and $\delta w_1' = \int \delta n' dt + \delta\epsilon'$. Hence we should have $\delta u_1 = \delta n'$, $\delta u_2 = \delta w_1'$, so that

$$\frac{d}{dt}\delta u_1 - \delta\frac{du_1}{dt} = \frac{d}{dt}\delta n', \quad \frac{d}{dt}\delta w_1' - \delta\frac{dw_1'}{dt} = \frac{d}{dt}\delta\epsilon'.$$

Now $\delta n'$, $\delta\epsilon'$ only contain the first power of the period as a large multiplier, and therefore the disturbing function does not contain it at all.* Hence all the terms produced by the use of this disturbing function are quite insensible.

We therefore proceed as with the secular accelerations, obtaining the variations of n, e, γ by solving the equations $\delta c_i = 0$ considered as functions of n, e, γ, n' in which $\delta n'$ is known. Then $\delta w_1 = \int \delta n dt$, etc. But this simply gives the primaries independent of the lunar angles, together with the secondaries arising from them, and these we have already obtained. The method therefore proves that for all other terms arising from long-period terms in $\delta V'$, $\delta\rho'$ we are to simply substitute the disturbed values of n', w_1' in the final expressions for the Moon's coordinates. The substitution of $\delta n'$ gives nothing sensible. Hence the statement.

353. *Computation of the secular variations.*—The values of δn, δb_2, δb_3 have been obtained in an earlier memoir.† With the adopted value of $\delta e'$, namely, $e_1' = -[\bar{6}\cdot 5968]n'$, I find

$$\delta n = +[\overline{11}\cdot 0294]nn't, \quad \delta b_2 = -[\overline{11}\cdot 8476]nn't, \quad \delta b_3 = +[\overline{11}\cdot 0746]nn't,$$

giving

$$\delta w_1 = +5''\cdot 82 t_c^2, \quad \delta w_2 = -38''\cdot 3 t_c^2, \quad \delta w_3 = +6''\cdot 46 t_c^2,$$

where t_c is the number of centuries from 1850·0.

To obtain the values of δe, $\delta\gamma$, we have

$$0 = \delta c_2 = \frac{dc_2}{dn}\delta n + \frac{dc_2}{de}\delta e + \frac{dc_2}{d\gamma}\delta\gamma + \frac{dc_2}{de'}\delta e',$$

with a similar equation for c_3. As δn is known, the values of δe, $\delta\gamma$ can be found from the formulæ (39) of Chap. X., by putting for δc_2, δc_3 the expressions

$$-e'\frac{dc_2}{de'}e_1't, \quad -e'\frac{dc_3}{de'}e_1't.$$

We find that $\delta e/e$, $\delta\gamma/\gamma$ are less than $10^{-8}t_c$, and therefore quite insensible.

* This is true in general. See *Trans. Amer. Math. Soc.*, vol. vi. p. 341.

† *Monthly Notices*, vol. lvii. pp. 342–349.

354. *Description of the tables.*—The portions of the detailed results selected for printing have been described in the preceding paragraphs; it is understood that *all* portions greater than $0''\cdot001$ have been computed and included in the collected results given in Section (viii) below.

The heading to each page sufficiently describes the contents in general. It will be noticed that $\psi + \alpha^\circ$ always denotes the argument of the primary, $\psi \pm l + \alpha^\circ$ being the argument of the principal secondary in longitude, and $\psi \pm \mathrm{F} + \alpha^\circ$ that in latitude. The angle ϖ'' is the longitude of perihelion of the particular planet considered, and l' the mean anomaly of the Earth's orbit.

The angle α is the same for all values of C in a given line until a new value is set down.

355. $\delta V = +0''\cdot001\, C \sin\{i'l' + i''(\mathrm{V} - \varpi'') + \theta + a^\circ\} = +0''\cdot001 \sin(\psi + a^\circ)$, *Venus.*

		$\theta=0$				$\theta=2\mathrm{D}$		$\theta=-2\mathrm{D}$	
		ψ		$\psi\pm l$		ψ		ψ	
i'	i''	a	C	a	C	a	C	a	C
0	−1	334	8	332	2	152	10		
1		330·9	354	330·9	72	150·9	111	150·9	50
2		48	12	90	3	319	2	175	4
3		28	1						
1	−2	122	13	122	5	302	17		
2		121·6	511	121·5	104	301·5	179	302	11
3		315·4	275	324	39	133·7	54	134	30
4		323	5					136	3
2	−3	91	1			272	2		
3		92·0	50	91	10	272	23	91	1
4		284·8	158	285	32	104·7	46	104	8
5		314·5	129	322	22	132	14	131	20
6		315	2					133	2
4	−4	64	14	64	3	244	9		
5		74	13	73	3	255	5		
6		287	17	288	3	106	4	106	1
5	−5	35	5			215	4		
6		48	3			226	2		
7		74	12	74	3	253	4		
8		271	19	274	4	90	3	90	2
6	−6	6	2			186	2		
9		256	2						
7	−7					156	1		

$$\delta V = +0''{\cdot}001\, C \sin\{i'l' + i''(\mathrm{V} - \varpi'') + \theta + \alpha^\circ\} = +0''{\cdot}001\, C \sin(\psi + \alpha^\circ).$$

		$\theta = l$				$\theta = -l$			
		ψ		$\psi - l$		ψ		$\psi + l$	
i'	i''	α	C	α (Same as for ψ)	C	α	C	α (Same as for ψ)	C
0	−1	332	1		4				
1		330·9	13		47	331	4		8
2		296			1	5			1
1	−2	122	2		7				
2		121·5	24		92	302	3		21
3		335	5		18	314	2		7
4						317			1
3	−3	91	3		13	273			4
4		285	6		21	103			3
5		309	2		5	312	2		7
4	−4	64	1		5	244			1
5		73			3				
6		287			2				
5	−5	35			2				
7		74			2				
8		271			1				

			$\theta = 2\mathrm{D} + l$		$\theta = -2\mathrm{D} - l$		$\theta = 4\mathrm{D} - l$		$\theta = -4\mathrm{D} + l$
			ψ	$\psi - l$	ψ	$\psi + l$	ψ	$\psi + l$	$\psi - l$
i'	i''	α	C	C	C	C	C	C	C
0	−1	152		2					
1		151	4	20	2	10	2	5	2
1	−2	302		3				1	
2		301	6	32		2	2	7	
3		133	2	10	1	6		2	1
3	−3	271		4					
4		105	2	8		1		2	
5		133		3		4			
4	−4	244		2					

$$\delta V = +0''\cdot 001\, C \sin\{i'l' + i''(\mathrm{V} - \varpi'') + \theta + a^\circ\} = +0''\cdot 001\, C \sin(\psi + a^\circ).$$

		$\theta = 2\mathrm{D} - l$			$\theta = -2\mathrm{D} + l$		
			ψ	$\psi + l$		ψ	$\psi - l$
i'	i''	a	C	C	a	C	C
0	−1	332	4	19			
1		330·9	41	214	330·9	16	91
2		300		2	358	2	9
1	−2	121	7	32			
2		121·4	72	359	122	5	13
3		313·3	19	100	313·3	9	55
4					316	1	6
2	−3	92		5			
3		91·0	10	47	272		3
4		284·7	19	91	284	2	13
5		312	5	26	311	7	37
6					313		3
4	−4	64	4	20	244		2
5		74	2	10			
6		286	1	7	286		2
5	−5	35	2	9			
6		52		4			
7		74	2	8			
8		271	1	5	271		4
6	−6	6	1	5			
7		18		1			
7	−7	336		3			
8	−8	307		2			

		$\theta = 2l$		$\theta = -2l$
			$\psi - l$	$\psi + l$
i'	i''	a	C	C
1	−1	151	3	1
2	−2	302	5	
3		134	1	
3	−3	272	1	

$$\delta V,\ \delta U = +0''\cdot001\, C \sin\{i'l' + i''(\mathrm{V} - \varpi'') + \theta + \alpha^\circ\} = +0''\cdot001\, C \sin(\psi + \alpha^\circ).$$

		$\theta = 2l - 2\mathrm{D}$			$= -2l + 2\mathrm{D}$		
			ψ	$\psi - l$		ψ	$\psi + l$
i'	i''	α	C	C	α	C	C
0	−1	159		4	332		5
1		330·9	3	77	330·9	4	87
2		2		3	108		1
1	−2	122		3	122		6
2		301	6	10	121·5	6	112
3		314	2	36	314	2	48
4		317		2			
3	−3	92·5	−3	59	91		12
4		284		11	285	2	33
5		311		19	312		16
6		313		1			
4	−4	64		4	64		4
5		254		3	74		3
6		286		2	286		3
5	−5	36		1	35		1
6		45		1			
7					74		2
8		270		2	270		3
8	−6	236		4			

		$\theta = 2\mathrm{F} - 2\mathrm{D}$			$\theta = -2\mathrm{F} + 2\mathrm{D}$		
			ψ	$\psi - \mathrm{F}$		ψ	$\psi + \mathrm{F}$
i'	i''	α	C	C	α	C	C
1	−1	331	1	11	331	1	11
2	−2	122	1	6	121	2	14
3		314		5	313		7
3	−3				91		1
4		284		2	285		4
5		311		3	313		2

356. $\delta V = +0''\cdot001\, C \sin\{i'l' + i''(\mathrm{M} - \varpi'') + \theta + a^\circ\} = +0''\cdot001\, C \sin(\psi + a^\circ)$, *Mars.*

		$\theta=0$				$\theta=2\mathrm{D}$		$\theta=-2\mathrm{D}$	
		ψ		$\psi\pm l$		ψ		ψ	
i'	i''	a	C	a	C	a	C	a	C
−1	1	53	30	53	7	232	7	232	3
0		9	5					196	1
−3	2	285	4	285	1	106	6		
−2		286·1	206	286·1	42	107·2	61	105	10
−1		72·8	271	81·4	71	251	29	249	32
0		299	3					265	3
−3	3	335	11	335	3	151	5		
−2		291·6	50	292	11	111	10	111	5
−4	4	194	3	194	1	5	2		
−3		164·8	51	165	11	347	15	342	3
−2		297·3	82	297	18	117	9	115	11
−1		303	1						
−4	5	215	7	215	1	31	3		
−3		170	25	170	5	349	5	349	2
−5	6					254	1		
−4		42	16	42	3	223	4		
−3		176	14	176	3	356	1	350	2
−5	7	93	5	93	1	270	2		
−4		48	13	48	3	226	2	228	1
−5	8	280	6			101	1		
−6	9	332	3			150	1		

		$\theta=l$			$\theta=-l$		
			ψ	$\psi-l$		ψ	$\psi+l$
i'	i''	a	C	C	a	C	C
−1	1	52		3			
−3	2	285		2			
−2		287	8	29	104		5
−1		67	2	8	69	3	11
−3	3	332		2			
−2		292	1	4			
−3	4	166	2	7	340		1
−2		293		2	296	1	4
−4	5	213		2			
−3		170		2			
−4	6	43		2			

$$\delta V = +0''{\cdot}001\, C \sin\{i'l' + i''(\mathrm{M} - \varpi'') + \theta + a^\circ\} = +0''{\cdot}001\, C \sin(\psi + a^\circ).$$

		$\theta = 2\mathrm{D} - l$			$\theta = -2\mathrm{D} + l$		
			ψ	$\psi + l$		ψ	$\psi - l$
i'	i''	a	C	C	a	C	C
−2	1	53		1			
−1		52	2	12	52		5
0					15		2
−3	2	286	2	10			
−2		287·0	23	117	285	3	15
−1		70·4	10	52	68·8	11	57
0					86		5
−3	3	331	2	10			
−2		291	3	18	291	1	8
−4	4	186		4			
−3		166	5	27	164		4
−2		297	3	15	295	4	20
−1					299		2
−4	5	212	1	6			
−3		169	2	9	169		4
−5	6	74		2			
−4		43	2	8	42		1
−3		175		2	172		4
−5	7	90		3			
−4		47		4	48		2
−5	8	282		3			
−6	9	329		2			

		$\theta = 2\mathrm{D} + l$			$\theta = -2\mathrm{D} - l$		$\theta = 4\mathrm{D} - l$	$\theta = -4\mathrm{D} + l$
			ψ	$\psi - l$	ψ	$\psi + l$	$\psi + l$	$\psi -$
i'	i''	a	C	C	C	C	C	C
−1	1	232		1				
−3	2	105		1				
−2		106	2	11		2	2	
−1		105	1	5	1	5	1	1
−2	3	111		2				
−3	4	347		3				
−2		117		2		2		

$$\delta V,\ \delta U = +0''{\cdot}001\, C \sin\{i'l' + i''(\mathrm{M} - \varpi'') + \theta + \alpha^\circ\} = +0''{\cdot}001\, C \sin(\psi + \alpha^\circ).$$

		$\theta = 2l - 2\mathrm{D}$			$\theta = -2l + 2\mathrm{D}$		
			ψ	$\psi - l$		ψ	$\psi + l$
i'	i''	α	C	C	α	C	C
−1	1	52		3	52		6
0		14		1			
−3	2	287		2	286		2
−2		286		12	286·9	3	42
−1		69	1	32	70	2	29
0		90		2			
−3	3				332		3
−2		291		6	291		9
−4	4	197		3	186		1
−3		165		3	166		10
−2		295		11	296		9
−4	5				212		2
−3		169		3	169		4
−5	6	74	−2	5			
−4		42		1	43		3
−3		173		2	174		1
−5	7				91		1
−4		48		2	47		2
−6	8	135	−1	3			
−5					281		1

		$\theta = 2\mathrm{F} - 2\mathrm{D}$		$\theta = -2\mathrm{F} + 2\mathrm{D}$		$\theta = 2l$	
		$\psi - \mathrm{F}$		$\psi + \mathrm{F}$		$\psi - l$	
i'	i''	α	C	α	C	α	C
−2	2	286	3	286	6	106	2
−1		67	4	70	4		
−3	4			166	1		
−2		295	1	297	1		

357. $\delta V = +0''\cdot 001\, C \sin\{i'l' + i''(\text{J} - \varpi'') + \theta + \alpha^\circ\} = +0''\cdot 001\, C \sin(\psi + \alpha^\circ)$.

Jupiter.

		$\theta=0$				$\theta=2\text{D}$		$\theta=-2\text{D}$	
		ψ		$\psi\pm l$		ψ		ψ	
i'	i''	α	C	α	C	α	C	α	C
−2	1	77	17	82	6	265	19		
−1		90·4	724	90·4	153	269·7	211	271	34
0		348·4	262	312·6	65	182	39	190·5	46
1		2	8	359	2			182	5
−3	2	186	7	185	2	5	10		
−2		182·7	208	182·5	45	2·2	104	183	2
−1		161·6	173	163·0	39	339·9	44	340	9
0		358	14	348	6	183	2		
3	3	102	9	105	2	290	7		
−2		187·0	43	186	9	7	21		
−1		173	23	174	5	351	5	352	1
−3	4	101	3			287	2		
−2		191	7	191	2	13	3		
−1		177	3						

			$\theta=l$		$\theta=-l$		
			ψ	$\psi-l$		ψ	$\psi+l$
i'	i''	α	C	C	α	C	C
−2	1	83	2	8			
−1		90·0	27	102	272		14
0		17	4	13	8	5	16
1					0		2
−3	2	185	1	4			
−2		182·4	14	56	3	2	14
−1		162	6	23	344		3
−3	3	107	1	4	281		1
−2		187	3	11	7		3
−1		173		3			
−3	4	104		1			
−2		191		1			

$$\delta V = +0''\cdot 001\, C \sin\{i'l' + i''(\mathrm{J} - \varpi'') + \theta + \alpha''\} = +0''\cdot 001\, C \sin(\psi + \alpha^{\circ}).$$

		$\theta = 2\mathrm{D} - l$			$\theta = -2\mathrm{D} + l$		
			ψ	$\psi + l$		ψ	$\psi - l$
i'	i''	α	C	C	α	C	C
−2	1	84	7	37			
−1		89·8	82	417	90·9	10	58
0		4·5	14	75	10·1	16	87
−3	2	185	4	19			
−2		182·3	44	214	3·2	1	6
−1		160·3	17	89	160	3	15
0		8		3	44		3
−3	3	109	3	15	276		1
−2		187·2	9	42	6		1
−1		171	2	11	179		2
−4	4	18		2			
−3		106		4			
−2		192	1	6			
−1		177		1			

		$\theta = 2\mathrm{D} + l$			$\theta = -2\mathrm{D} - l$		
			ψ	$\psi - l$		ψ	$\psi + l$
i'	i''	α	C	C	α	C	C
−2	1	265		4			
−1		270	7	39	272	1	7
0		181	1	7	191	2	9
−3	2	5		2			
−2		2	4	19			
−1		340	1	8	340		2
−3	3	291		1			
−2		7		4			
−1		351		1			

		$\theta = 4\mathrm{D} - l$			$\theta = -4\mathrm{D} + l$		
			ψ	$\psi + l$		ψ	$\psi - l$
i'	i''	α	C	C	α	C	C
−2	1	267		1			
−1		269	3	9	272		1
0		181		2	193		2
−2	2	2	1	4			
−1		340		2			

$$\delta V = +0''{\cdot}001\, C \sin\{i'l' + i''(\mathrm{J} - \varpi'') + \theta + \alpha^\circ\} = +0''{\cdot}001\, C \sin(\psi + \alpha^\circ).$$

		$\theta = 2l - 2\mathrm{D}$			$\theta = -2l + 2\mathrm{D}$		
			ψ	$\psi - l$		ψ	$\psi + l$
i'	i''	α	C	C	α	C	C
−2	1	72		9	84		8
−1		90·6		47	89·9	8	150
0		8·9	2	46	6·2	2	44
1		1		3			
−3	2	187		1	185		3
−2		182·8	53	338	182·4	3	53
−1		158		12	161	2	33
0		33		1	15		1
−3	3	100		2	108		3
−2		5·9	94	130	187		11
−1		170		2	172		4
−2	4	11	1	4	192		2

		$\theta = 2l$			$\theta = -2l$		
			ψ	$\psi - l$		ψ	$\psi + l$
i'	i''	α	C	C	α	C	C
−1	1	270		6			
0		187		1	187		1
−2	2	3		3			
−1		341		1			

		$\theta = 2\mathrm{F} - 2\mathrm{D}$			$\theta = -2\mathrm{F} + 2\mathrm{D}$		
			ψ	$\psi - \mathrm{F}$		ψ	$\psi + \mathrm{F}$
i'	i''	α	C	C	α	C	C
−2	1				84		1
−1		91	1	10	90	2	19
0		11		6	2		6
−2	2				182		7
−1		161		2	160		4
−2	3				187		1

358. $\delta V = +0''\cdot 001\, C \sin \{i'l' + i''(\mathrm{S} - \varpi'') + \theta + \alpha^\circ\} = +0''\cdot 001\, C \sin (\psi + \alpha^\circ)$,
Saturn.

		$\theta = 0$							
		ψ		$\psi \pm l$		$\theta = 2\mathrm{D}$		$\theta = -2\mathrm{D}$
i'	i''	α	C	α	C	α	C	C
−1	1	169·4	42	169	8	350	12	2
0		356·5	45	354	16	180	5	5
−2	2	339	8	339	1	159	4	
−1		337	11	337	3	156	3	
0		117	3					
2	3	340	2					
−1		341	2					

			$\theta = l$		$\theta = -l$	$\theta = 2\mathrm{D} - l$		$\theta = -2\mathrm{D} + l$		
			ψ	$\psi - l$	$\psi + l$	ψ	$\psi + l$		ψ	$\psi - l$
i'	i''	α	C	C	C	C	C	α	C	C
−2	1	169					2			
−1		169	2	6	−1	5	24	169		3
0		0		2	2	2	10	8	2	10
−2	2	339		2		2	9			
−1		337		2		1	6			
−2	3	340					2			

		$\theta = 2\mathrm{D} + l$		$\theta = -2\mathrm{D} + l$	$\theta = 2l - 2\mathrm{D}$		$\theta = -2l + 2\mathrm{D}$
		$\psi - l$		$\psi - l$	$\psi - l$		$\psi + l$
i'	i''	α	C	C	α	C	C
−1	1	349	2		169	2	11
0		180	1	1	0	6	6
−2	2				341	5	2
−1					337		2
−2	3				334	1	

359. $\delta V = +0''\cdot 006 \sin (l' + 347^\circ)$, *all planets.*

Section (viii). *Final Results for the Indirect Action.*

360. *Tests.*—No general method of testing the work appeared to be available, but various peculiarities of the solution very much aided in the avoidance of the kind of error which is most likely to occur—one running through a whole series of terms.

In order to try and abbreviate the work, the two terms with the same argument of the form $a \sin \psi + b \cos \psi$ in the disturbing function were combined into a single argument of the form $A \cos (\psi + \alpha)$ as early as possible. This might have become very disadvantageous in the final process, where terms with different values of A, α had to be added together. It was, however, obvious that for nearly all terms with the same ψ, the angle α should be nearly the same; and the cases where this was not to be expected were evident. This approximate equality of the angles therefore served as a test.

Again, let us consider the terms with arguments independent of the lunar angles. They are formed of the primaries with arguments ϕ and the secondaries with arguments $(\phi + l) - l$, $(\phi - l) + l$, $(\phi + 2\mathrm{D} - l) - 2\mathrm{D} + l$, etc. When the final addition was made, it was found that *the sum of the secondaries was always small compared with the primary*, unless the primary was a term of very long period—an exception of rare occurrence and easily noticed. Consideration of the peculiarities of the method of variation of the constants showed that this must necessarily be the case.* This fact furnished a full test of the principal terms whose primaries have the arguments $\phi \pm l$, since these secondaries are the largest terms arising from those arguments; it was also a partial test of the terms with arguments $\phi \pm (2\mathrm{D} - l)$ in the disturbing function.

The final terms containing $\phi + 2\mathrm{D}$ arise chiefly from the primaries with arguments $\phi + 2\mathrm{D}$ and the secondaries with arguments $(\phi + 2\mathrm{D} - l) + l$. In general, the latter are very nearly half the former; this can also be shown to be a consequence of the theory. This tests the terms with arguments $2\mathrm{D}$, $2\mathrm{D} - l$ in the disturbing function.

The only important terms not tested by these methods are those with arguments $\phi \pm (2l - 2\mathrm{D})$, all of which have periods much longer than the month. The resulting terms in the longitude which have the largest coefficients are those with arguments $\phi \pm (l - 2\mathrm{D})$.

The existence of these tests raises a doubt as to whether the variation of arbitrary constants is the best method for treating the numerous short-period terms. Possibly a direct method might be more simple. It would certainly have the advantage of finding the changes in the coordinates directly, and of avoiding the formation of derivatives with respect to n. It might be advisable to use a direct method for most of the terms, and to use the variation of the elements only for those of long period.

361. *Arrangement.*—The various terms with the same period which arise from the substitution of the elements in the Moon's coordinates have been collected into one term; in some cases there were as many as eight such terms.

* See *Monthly Notices*, vol. lxviii. p. 166.

As with the direct inequalities, the terms are divided into classes, those added to the coordinates and those added to the elements. The selection is somewhat arbitrary, and it largely depends on convenience for the formation of the tables of the Moon's motion which will be a natural sequence of this work. There are no terms which seem to require the other class used with the direct terms—those partly added to the coordinates and partly to the elements. In other respects the arrangement is quite the same.

The angles in the preceding section were the mean anomalies of the planets; they are changed here to mean longitudes to facilitate addition with the direct terms.

The terms given are those equal to or greater than $0''\cdot001$, with the exception of a few (*e.g.* in the latitude due to Mars) for which there are no portions due to other causes. But all terms greater than $0''\cdot002$ are included, except one or two of very long period, noted above.

In the terms added to the elements, t_c represents the number of centuries reckoned from 1850·0.

The two terms added to $n't+\epsilon'$ are to be included in the arguments $D = nt+\epsilon-n't-\epsilon'$, and $l' = n't+\epsilon'-\varpi'$; and the secular part of ϖ' is to be included in the latter argument.

All terms given at the end of Chap. IX. which contain the arguments $\pm il'$ (i positive) are to receive the factor $1+i'\delta e'/e' = 1-\cdot00248i't_c$.

362. $\delta V = +0''{\cdot}001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T}-\mathrm{V}) + \alpha^\circ\}$, *Venus.*

$\theta = 0$

j	i	α	C
0	1	0·0	344
	2	179·8	507
	3	179·6	50
	4	180	14
	5		5
	6		1
1	−2	259	14
	−1	77	8
	1	323	10
	2	273·3	272
	3	271·8	155
	4	90	14
	5	93	3
2	1	216	1
	2	180	5
	3	201·1	129
	4	203	18
	5	19	12
3	3	92	2
	5	115	19
	6	120	2

$\theta = 2\mathrm{D}$

j	i	α	C
0	−4	180	2
	−3	0	2
	−2	0	10
	−1	180	38
	1	0·0	84
	2	179·6	144
	3	178	17
	4	180	8
	5		5
	6		3
	7		2
	8		2
1	1	232	3
	2	271	40
1	3	271	37
	4	89	5
	5	103	2
	6	92	1
−1	−3	269	6
	−2		23
	−1	254	5
	1	101	8
	2	281	13
	3		3
2	3	199	11
	4	202	3
	5	20	4
−2	−4	338	1
	−3	342	15
	−2	7	3
3	5	116	2
−3	−5	64	2
	−3	91	1

$\theta = l$

j	i	α	C
0	−4	0·0	3
	−3		8
	−2		90
	−1	180·0	61
	1	0·0	61
	2	180·0	112
	4		4
1	−2	258	4
	−1	79	1
	1	27	3
	2	271	33
	3	272	34
	4	91	3
−1	−4	89	3
	−3	268	27
	−2	269	33
−1	−1	151	3
	1	102	3
	2	282	7
2	3	209	20
	4	204	3
	5	19	3
−2	−5	161	3
	−4	336	3
	−3	331	21
3	5	115	3
−3	−5	65	3

$\theta = 2\mathrm{D} - l$

j	i	α	C
0	−5	0·0	1
	−4		4
	−3		58
	−2	180·0	15
	−1		87
	1	0·0	120
	2	179·6	167
	3	178	21
	4	180	8
	5		3
	6		1
1	−2	259	3
	−1	252	4
	1	37	1
	2	271·4	65
	3	271·9	49
	4	90	5
−1	−5	90	1
	−4	270	3
	−3	269	10
	−2	268·1	43
	−1	251	5
	1	101	9
	2	281	13

$$\delta V = +0''\cdot 001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T}-\mathrm{V}) + \alpha^\circ\}. \quad \textit{Venus.}$$

$\theta = 2\mathrm{D} - l$

j	i	α	C
2	3	200	19
	4	202	4
	5	19	4
−2	−6	340	4
	−4	338	2
	−3	340	24
	−2	6	3
3	5	114	4
−3	−5	66	2
	−3	91	1

$\theta = 2\mathrm{D} + l$

j	i	α	C
0	−1	180	4
	1	0	10
	2	180	15
	3		3
	4		1
1	2	271	5

$\theta = 2\mathrm{D} + l$

j	i	α	C
1	3	271	4
−1	−2	269	3
	1	101	1
	2	281	2
2	3	199	2
−2	−3	341	2

$\theta = 4\mathrm{D} - l$

j	i	α	C
0	1	180	6
	2	0	7
1	2	92	1
	3		1

$\theta = 2l$

j	i	α	C
0	−2	0	7
	−1	180	5
	1	0	5
	2	180	8

$\theta = 2l$

j	i	α	C
1	2	273	3
	3	272	2
−1	−3	268	2
	−2	267	3
2	3	202	1
−2	−3	238	1

$\theta = 2l - 2\mathrm{D}$

j	i	α	C
0	−2	0	7
	−1	180	5
	1	0	2
	2		8
	3		3
1	2	272	2
−1	−3	268	3
	−2		3
2	6	200	3

$$\delta V = +0''\cdot 001\, C \sin(\theta + \phi).$$

$\phi = 4\mathrm{T} - 3\mathrm{V} + 272^\circ\cdot 1$

θ	C
$l - \mathrm{D}$	51
$-\mathrm{D}$	32
$2l - \mathrm{D}$	2
$\mathrm{D} + 180^\circ$	7
$-l - \mathrm{D}$	3

$\phi = 5\mathrm{T} - 3\mathrm{V} + 216^\circ$

θ	C
w_3	19
$w_3 \pm l$	3
$w_3 + 2\mathrm{F}$	4
$-w_3 - 2\mathrm{F}$	2

363. $\delta V = +0''\cdot 001\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M}-\mathrm{T}) + \alpha^\circ\}$, *Mars.*

$\theta = 0$

j''	i	α	C
0	1	180	30
	2	180·2	203
	3	356	11
	4	342	3
1	−3	260	6

$\theta = 0$

j''	i	α	C
1	0	36	5
	1	227·8	269
	2	212·5	48
	3		52
	4	320	9

$\theta = 0$

j''	i	α	C
2	0	183	2
	2	245·0	81
	3		25
	4	244	16
	5	62	5

$$\delta V = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M}-\mathrm{T}) + a^\circ\},\ \textit{Mars}.$$

$\theta = 0$

j''	i	a	C
3	1	150	1
	3	277	14
	4	276	14
	5	275	6
	6	94	3

$\theta = 2D$

j''	i	a	C
0	−2	0	5
	−1		2
	1	180	4
	2	181	44
	3	0	5
	4	328	2
1	1	224	23
	2	212	6
	3	214	8
	4	37	3
	5	16	1
−1	−3	327	1
	−2	328	3
	−1	317	23
	0	139	1
	2	280	1
	3		3
2	2	244	5
	3		4
	4	246	4
	5	59	1
−2	−4	296	1
	−3		2
	−2	297	8
	0	40	2
3	3	275	1
	4	276	2
	5	277	2
	6	90	1
−3	−4	264	1
	−3	264	2
	−1	34	2

$\theta = l$

j''	i	a	C
0	−3	180	3
	−2	0	38
	−1		6
	1	180	7
	2		43
	3	0	3
1	−3	261	1
	1	224·2	66
	2	212	11
	3	213	13
	4	30	1
	5	196	1
−1	−4	150	1
	−3	330	9
	−2	327	9
	−1	304·2	67
	3	279	1
2	2	246	16
	3	245	5
	4	244	3
	5		1
−2	−5	296	1
	−4		3
	−3	295	5
	−2	294	17
3	3	277	3
	4	276	3
−3	−4	264	3
	−3	263	2

$\theta = 2D - l$

j''	i	a	C
0	−4	195	3
	−2	0	11
	−1	0	3
	1	180	8
	2	181·0	61
	3	353	5
	4	329	1
1	−3	259	2
	1	220	31
	2	212	11
	3	214	14
	4	27	3
−1	−5	164	5
	−3	327	2
	−2	328	6
	−1	320	35
	0	139	1
	3	280	4
2	2	244	11
	3		6
	4	245	5
	5	60	1
−2	−6	309	3
	−4	296	1
	−3		3
	−2	297	14
3	3	275	1
	4		2
	5	276	1
−3	−4	264	2
	−3	266	2

$\theta = 2D + l$

j''	i	a	C
0	2	180	6
	−2	0	1
1	1	82	3
	2	212	1
	3	214	2
−1	−1	93	3

$$\delta V = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M}-\mathrm{T}) + \alpha^\circ\},\ \textit{Mars.}$$

$\theta = 2\mathrm{D} + l$

j''	i	α	C
2	2	245	1
−2	−2	297	1

$\theta = 4\mathrm{D} - l$

j''	i	α	C
0	2	0	2

$\theta = 2l$

j''	i	α	C
0	−2	0	3
	2	180	3
1	1	232	3
−1	−1	308	3

$\theta = 2l - 2\mathrm{D}$

j''	i	α	C
0	−2	0	4
1	1	87	1
	5	196	2
−1	−1	98	2
2	6	51	1

364. $$\delta V = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{J} + i(\mathrm{J}-\mathrm{T}) + \alpha^\circ\},\ \textit{Jupiter.}$$

$\theta = 0$

j''	i	α	C
0	1	178·9	712
	2	359·6	200
	3	7	10
1	−3	257	6
	−2	274	18
	0	336·3	259
	1	238·0	170
	2	352	44
	3	355	4
2	−1	250	10
	0	334	14
	1	238	25
	2	344	6
3	1	230	3

$\theta = 2\mathrm{D}$

j''	i	α	C
0	−3	0	1
	−2	180	25
	−1	1	31
	1	178·5	167
	2	359·2	87
	3	13	7
	4	12	2
1	0	349	29
	1	237	35
	2	352	15
	3	358	2

$\theta = 2\mathrm{D}$

j''	i	α	C
−1	−2	9	10
	−1	303	6
	0	184	35
	2	273	9
	3	102	6
2	0	351	1
	1	236	5
	2	345	3
−2	−1	288	1
	0	200	3
	1	110	6
3	1	230	1

$\theta = l$

j''	i	α	C
0	−3	173	2
	−2	180	40
	−1	1·0	136
	1	179·0	150
	2	180	19
	3	25	3
1	−2	274	6
	0	298·5	71
	1	239	40
	2	351	38
	3	257	1
−1	−2	188	7
	−1	301	36

$\theta = l$

j''	i	α	C
−1	0	241·5	72
	2	273	6
2	−1	250	2
	0	326	7
	1	238	5
	2	344	2
−2	−2	196	2
	−1	302	5
	0	214	7
	1	290	2

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−3	175	2
	−2	180·4	333
	−1	1	44
	1	178·4	211
	2	359·2	89
	3	14	6
1	−3	263	1
	−2	279	9
	0	5·5	56
	1	237·0	46
	2	352	20
−1	−2	9·1	130
	−1	301	13
	0	174·2	60
	2	273	16

$$\delta V = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{J} + i(\mathrm{J}-\mathrm{T}) + \alpha^\circ\},\ \textit{Jupiter.}$$

$\theta = 2\mathrm{D} - l$

j''	i	α	C
−1	3	102	7
2	0	351	1
	1	237	6
	2	344	3
−2	−2	196	4
	−1	305	2
	0	171	1
	1	291	3

$\theta = 2\mathrm{D} + l$

j''	i	α	C
0	−1	1	5
	1	178	21
	2	359	8
	3	14	1
1	0	353	4
	1	237	5
	2	352	3
−1	−1	303	1
	0	182	4

$\theta = 2\mathrm{D} + l$

j''	i	α	C
−1	2	273	3
	3	102	1

$\theta = 4\mathrm{D} - l$

j''	i	α	C
0	1	358	9
	2	179	5
1	0	172	1
	1	57	1
−1	0	2	1

$\theta = 2l$

j''	i	α	C
0	−2	180	3
	−1	2	11
	1	178	12
	2	0	3
1	0	293	5
	1	239	3
	2	171	1
−1	−1	301	3
	0	247	5

$\theta = 2l - 2\mathrm{D}$

j''	i	α	C
0	−2	180	5
	−1	2	11
	1	0	3
	2	179·7	53
1	2	170·9	94
−1	−1	302	2
	0	186	2
2	2	164	1

$\theta = 2l - 4\mathrm{D}$

j''	i	α	C
1	2	171	1

$\theta = 2\mathrm{F} - 2\mathrm{D}$

j''	i	α	C
0	−1	0	2
	1	180	1

$\theta = w_3$

j''	i	α	C
1	0	0	3
2	0	168	6

365. $\delta V = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{S} + i(\mathrm{S}-\mathrm{T}) + \alpha^\circ\}$, *Saturn.*

$\theta = 0$

j''	i	α	C
0	1	179·6	42
	2	0	8
1	0	266·6	45
	1	257	13
	2	270	2
2	0	297	3
	1	171	2

$\theta = 2\mathrm{D}$

j''	i	α	C
0	−1	6	1
	1	180	10
	2	0	5
1	0	270	4

$\theta = 2\mathrm{D}$

j''	i	α	C
1	1	257	3
	2	270	2
−1	0	255	4
	2	280	2

$\theta = l$

j''	i	α	C
0	−2	180	1
	−1	0	6
	1	180	10
1	0	265	15
	1	257	2
−1	−1	283	3
	0	275	15

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−2	180	5
	−1	0	2
	1	180	14
	2	0	4
1	0	271	6
	2	257	3
−1	−2	277	1
	0	267	6

$\theta = 2\mathrm{D} + l$

j''	i	α	C
0	1	180	2

366. *All planets.*

$$\delta V = -0''\cdot024 \sin(l' - 5^\circ) - 0''\cdot004 \sin(l' \pm l - 5^\circ) - 0''\cdot002 \sin(l' + l - 2\mathrm{D} - 5^\circ).$$

Latitude.

367. $\delta U = +0''\cdot001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T} - \mathrm{V}) + \alpha^\circ\} = +0''\cdot001\, C \sin\psi$, *Venus.*

$\theta = \pm\mathrm{F}$

j	i	α	C
0	1	0	4
	2	180	7
1	2	273	6
	3	272	2
2	3	201	2

$\theta = -\mathrm{F} + 2\mathrm{D}$

j	i	α	C
0	−2	0	7
	−1	180	14
	1	0	16
	2	180	23
	3		2
1	2	271	9
	3	272	6
−1	−3	268	2
	−2	269	6
	1	101	1
	2	280	1
2	3	199	3
−2	−3	341	5

$\theta = \mathrm{F} + 2\mathrm{D}$

j	i	α	C
0	−1	180	2
	1	0	5
	2	180	9
	3		1
1	2	271	2
	3	272	2
−1	−2	268	1
	1	101	1
	2	280	1
2	3	199	1
−2	−3	341	2

$\theta = \pm\mathrm{F} \pm l$

j	i	α	C
0	1	0	2
	2	180	4
1	2	271	1
	3	272	1
2	3	199	1

$\theta = -\mathrm{F} + 2\mathrm{D} + l$

j	i	α	C
0	2	0	1

$\theta = \mathrm{F} + 2\mathrm{D} + l$

j	i	α	C
0	1	0	1
	2	180	2

$\theta = \pm\mathrm{F} + 2\mathrm{D} - l$

j	i	α	C
0	−3	0	3
	−1		3
	1		6
	2	180	8
1	2	271	3
	3		2
−1	−2	269	2
2	3	199	1

ψ	C
$w_1 + 5\mathrm{T} - 3\mathrm{V} + 215^\circ\cdot6$	77
$w_1 - 5\mathrm{T} + 3\mathrm{V} + 337^\circ$	30
$w_1 - 2\mathrm{D} + 5\mathrm{T} - 3\mathrm{V} + 36^\circ$	3
$w_1 + l + 5\mathrm{T} - 3\mathrm{V} + 216^\circ$	4
$w_1 - l + 5\mathrm{T} - 3\mathrm{V} + 36^\circ$	4
$w_1 + 8\mathrm{T} - 5\mathrm{V} + 125^\circ$	3
$w_1 - 8\mathrm{T} + 5\mathrm{V} + 67^\circ$	7

368. $\delta U = +0''\cdot001\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M} - \mathrm{T}) + \alpha^\circ\}$, *Mars.*

$\theta = \pm\mathrm{F}$

j''	i	α	C
0	2	180	3

$\theta = \pm\mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	−2	0	3

$\theta = \mathrm{F} - 2\mathrm{D}$

j''	i	α	C
0	−2	0	8
	2	180	3
1	1	223	5
−1	−1	316	5

$\theta = w_1$

j''	i	α	C
2	2	269	2

369. $\delta U = +0''{\cdot}001\, C \sin\{\theta + j''\mathrm{J} + i(\mathrm{J}-\mathrm{T}) + \alpha^\circ\} = +0''{\cdot}001\, C \sin\psi$, *Jupiter.*

$\theta = \pm \mathrm{F}$

j''	i	α	C
0	1	180	9
1	0	48	8

$\theta = -\mathrm{F} + 2\mathrm{D}$

j''	i	α	C
0	−2	0	3
	−1		6
	1	180	29
	2	0	12
1	0	350	7
	1	237	6
	2	352	2
−1	−1	303	2
	0	181	4
	2	273	3
	3	102	1

$\theta = \mathrm{F} + 2\mathrm{D}$

j''	i	α	C
0	−2	0	4
	−1	180	3
	1	180	8
	2	0	3
1	0	350	1
	1	237	2
	2	352	2
−1	0	0	2
	2	273	2
	3	102	1

$\theta = \pm \mathrm{F} + l$

j''	i	α	C
0	−2	0	2
	−1		6
	1	180	8
	2		1
1	0	301	4
	1	239	2
	2	350	1
−1	−1	301	2
	0	240	4

$\theta = \pm \mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	−2	180	4
	−1	0	11
	1	180	2
	2	0	15
1	0	357	2
	2	171	6
−1	−1	302	2
	0	186	2

$\theta = \pm \mathrm{F} + 2l - 2\mathrm{D}$

j''	i	α	C
0	2	357	2
1	2	354	4

ψ	C
$w_1 + \mathrm{J}$	5
$w_1 + 2\mathrm{J} + 168^\circ$	35
$w_1 - 2\mathrm{J} + 24^\circ$	18
$w_1 + 3\mathrm{J} + 156^\circ$	2
$w_1 + l + 2\mathrm{J} + 168^\circ$	2
$w_1 - l + 2\mathrm{J} + 348^\circ$	2

370. *Terms added to the Elements.*

$$\delta w_1 = +0''{\cdot}001\, C \sin\psi + 5''{\cdot}82 t_c^2$$

ψ	C
$13\mathrm{T} - 8\mathrm{V} + 313^\circ{\cdot}8$	234
$l + 16\mathrm{T} - 18\mathrm{V} + 331^\circ$	6*
$\mathrm{Q} - 4\mathrm{T} + 239^\circ$	3
$8\mathrm{M} - 4\mathrm{T} + 310^\circ$	3
$9\mathrm{M} - 5\mathrm{T} + 305^\circ$	8
$10\mathrm{M} - 6\mathrm{T} + 306^\circ$	2
$11\mathrm{M} - 6\mathrm{T} + 335^\circ$	6
$13\mathrm{M} - 7\mathrm{T} + 19^\circ$	6
$15\mathrm{M} - 8\mathrm{T} + 43^\circ$	26
$17\mathrm{M} - 9\mathrm{T} + 63^\circ$	4
$w_3 + 276^\circ{\cdot}2$	289

$$\delta(w_1 - w_2) = +16'' t_c + 44''{\cdot}1 t_c^2 + 1''{\cdot}129 \sin(w_3 + 276^\circ{\cdot}2)$$

$$\delta w_3 = +5'' t_c + 6''{\cdot}46 t_c^2 + 15''{\cdot}59 \sin(w_3 + 276^\circ{\cdot}2)$$

$$\delta\gamma = +0''{\cdot}698 \cos(w_3 + 96^\circ{\cdot}2)$$

$$\delta(n't + \epsilon') = +1''{\cdot}89 \sin(13\mathrm{T} - 8\mathrm{V} + 134^\circ) + 0''{\cdot}20 \sin(15\mathrm{M} - 8\mathrm{T} + 216^\circ)$$

$$\delta\varpi' = +0^\circ{\cdot}32 t_c$$

$$\delta e' = -{\cdot}00248 e' t_c$$

CHAPTER XIII.

ACTION OF THE FIGURES OF THE EARTH AND MOON.

Section (i). *The Disturbing Function for the Figure of the Earth.*

371. Let A, B, C, I be the moments of inertia of the Earth's mass about its three principal axes at the centre of mass and about the line joining its centre of mass with that of the Moon; E, M the masses of the Earth and Moon. Then it is well known that the disturbing function R is given, with sufficient approximation, by

$$R = (E+M)\frac{\mathrm{A}+\mathrm{B}+\mathrm{C}-3\mathrm{I}}{2r^3E} \qquad (1),$$

since the next term of R will have an additional factor of the order 60^{-2}, this being the approximate ratio of the square of the radius of the Earth to the distance of the Moon. It is true that there is a term with a factor of the order 60^{-1}, but this term is exceedingly minute, owing to the approximate symmetry of the Earth about its principal axes.

Let V, U, α, δ be the longitude on the ecliptic, the latitude, the right ascension reckoned from the A-axis, and the declination of the Moon; ψ the precession, ϵ_1 the obliquity of the ecliptic. If P be the pole of the ecliptic, Q that of the Earth's equator, the parts of the spherical triangle PQM are:

$$PE = \epsilon_1, \quad QM = 90^\circ - \delta, \quad MP = 90^\circ - U, \quad QPM = 90^\circ - V - \psi,$$

and therefore

$$\sin\delta = \cos\epsilon_1 \sin U + \sin\epsilon_1 \cos U \sin(V+\psi) \qquad (2).$$

Also

$$\mathrm{I} = \mathrm{A}\cos^2\alpha\cos^2\delta + \mathrm{B}\sin^2\alpha\cos^2\delta + \mathrm{C}\sin^2\delta,$$

so that

$$\mathrm{A}+\mathrm{B}+\mathrm{C}-3\mathrm{I} = 3\left(\mathrm{C}-\frac{\mathrm{A}+\mathrm{B}}{2}\right)\left(\tfrac{1}{3}-\sin^2\delta\right) - \tfrac{3}{2}(\mathrm{A}-\mathrm{B})\cos 2\alpha\cos^2\delta \qquad (3).$$

The second term of this is quite negligible: its principal arguments have daily mean motions of the order 3.10^6 seconds, and $\mathrm{A}-\mathrm{B}$ is known to be very small compared with $\mathrm{C}-\frac{1}{2}(\mathrm{A}+\mathrm{B})$. Hence

$$R = (E+M)\frac{a^2\mu}{r^3}\left(\tfrac{1}{3}-\sin^2\delta\right), \quad \text{where } a^2\mu = \frac{3}{2E}\left(\mathrm{C}-\frac{\mathrm{A}+\mathrm{B}}{2}\right) \qquad (4).$$

372. *Transformation.*—Since

$$\rho \cos (V - \mathrm{T}) \pm \iota\rho \sin (V - \mathrm{T}) = u,\ s, \qquad r \sin U = z, \qquad r \cos U = \rho,$$

equation (2) gives

$$r \sin \delta = z \cos \epsilon_1 + \frac{1}{2\iota} \sin \epsilon_1 (ue^{(\mathrm{T}+\psi)\iota} - se^{-(\mathrm{T}+\psi)\iota}),$$

$$\tfrac{1}{3}r^2 - r^2 \sin^2 \delta = (\tfrac{1}{3}r^2 - z^2)(1 - \tfrac{3}{2}\sin^2 \epsilon_1) + \tfrac{1}{4}\sin^2 \epsilon_1 (u^2 e^{2(\mathrm{T}+\psi)\iota} + s^2 e^{-2(\mathrm{T}+\psi)\iota}) - \frac{z}{2i} \sin 2\epsilon_1 (ue^{(\mathrm{T}+\psi)\iota} - se^{-(\mathrm{T}+\psi)\iota}).$$

The last two terms are the real parts of $\frac{1}{2}\sin^2\epsilon_1 \,.\, u^2 e^{2(\mathrm{T}+\psi)\iota}$, $zi \,.\, \sin 2\epsilon_1 \,.\, ue^{(\mathrm{T}+\psi)\iota}$. Hence R is equal to the real part of

$$\tfrac{1}{4}n'^2 \mathrm{a}^2 \left[\mu_1 \left\{ \left(\frac{a}{r}\right)^3 \frac{a^2}{\mathrm{a}^2} - 3\left(\frac{a}{r}\right)^5 \frac{z^2}{\mathrm{a}^2} \right\} + \mu_3 \left(\frac{a}{r}\right)^5 \frac{u^2}{\mathrm{a}^2} e^{2(\mathrm{T}+\psi)\iota} \Big) + \mu_2 \left(\frac{a}{r}\right)^5 \frac{u \cdot zi}{\mathrm{a}^2} e^{(\mathrm{T}+\psi)\iota} \right] \quad . \quad . \quad (5),$$

where

$$\mu_1 = \tfrac{4}{3}\frac{\mu}{m^2}(1 - \tfrac{3}{2}\sin^2 \epsilon_1), \qquad \mu_3 = \frac{2\mu}{m^2}\sin^2 \epsilon_1, \qquad \mu_2 = \frac{4\mu}{m^2}\sin 2\epsilon_1 \quad . \quad . \quad . \quad (6),$$

μ being given in (4).

The values of the Moon functions have been given in Sect. (v), Chap. X. All the terms have the factor μ, which is treated as a small quantity of the first order, and we should properly put $\epsilon_1 = \text{const.}$ and $\psi = 0$. But it is convenient to retain the mean motion of ψ, as this motion affects the arguments to a slight degree, and is retained without any increase of labour. The μ_i are then constants which take the place of m''/m' in the equations of variations. In forming j_1, it is to be remembered that μ_i contains n^2.

Section (ii). *Numerical Results.*

373. *Adopted values of the constants.*—I take

$$\epsilon_1 = 23^\circ\ 27'\ 32'', \quad \text{daily motion of } \psi = +0''\cdot 14, \quad \mu = +[\bar{7}\cdot 6658] \quad . \quad . \quad . \quad (7),$$

giving

$$\mu_1 = +[\bar{5}\cdot 9251], \quad \mu_2 = +[\bar{4}\cdot 3836], \quad \mu_3 = +[\bar{5}\cdot 4191] \quad . \quad . \quad . \quad . \quad (8).$$

The only one of these constants of which the value is doubtful, within the limits of accuracy required here, is μ. It will ultimately be determined by the coefficient of the argument $w_1 + \psi$ in latitude, the principal term arising from the figure of the Earth. I adopt a value here to correspond with that marked (β) in my paper "On the Degree of Accuracy, etc." *; this is obtained by comparing Hansen's observational value with Hill's theoretical values (which closely agree with those obtained by me) for the coefficient in question.

374. *Final results.*—I omit terms whose coefficients are less than $0''\cdot 003$, and obtain, for the terms in longitude,

$$\delta V = +0''\cdot 020 \sin (2\mathrm{D} - l) + 0''\cdot 004 \sin (2\mathrm{F} - l) - 0''\cdot 038 \sin (2w_1 + 2\psi - 2\mathrm{F});$$

* *Monthly Notices,* vol. lxiv. p. 531.

in latitude,

$$\delta U = +0''\cdot 083 \sin(2w_1 + 2\psi - F) - 0''\cdot 003 \sin(2w_1 + 2\psi - F - 2D)$$
$$\pm 0''\cdot 005 \sin(2w_1 + 2\psi - F \pm l) - 0''\cdot 017 \sin(w_1 + \psi) - 0''\cdot 007 \sin(w_1 + \psi - 2D);$$

and, added to the elements,

$$\delta w_1 = +7''\cdot 317 \sin(w_3 + \psi), \quad \delta w_2 = +641'' t_c - 2''\cdot 092 \sin(w_3 + \psi), \quad \delta w_3 = -600'' t_c + 96''\cdot 69 \sin(w_3 + \psi),$$
$$\delta n = -0''\cdot 009 \cos(w_3 + \psi), \quad \delta e = +0''\cdot 002 \cos(w_3 + \psi), \quad \delta\gamma = -4''\cdot 351 \cos(w_3 + \psi),$$

of which δn, δe may be neglected.

The principal term in latitude which results from these values is $-8''\cdot 355 \sin(w_1 + \psi)$.

Section (iii). *The Action of the Figure of the Moon.*

375. *The Disturbing Function* is of the same form as that for the figure of the Earth. Let α' denote the longitude from the A′-axis on the Moon's equator of the projection of r on this plane, and δ' the inclination of r to the same plane. Then if A′, B′, C′, I′ be the moments of inertia, the C′-axis being that perpendicular to the Moon's equator, and I′ the moment of inertia about r, we have, as in § 371,

$$A' + B' + C' - 3I' = 3\left(C' - \frac{A' + B'}{2}\right)(\tfrac{1}{3} - \sin^2\delta') + \tfrac{3}{2}(B' - A') \cos 2\alpha' \cos^2\delta'. \quad . \quad . \quad (9).$$

376. *Transformation.*—Now the Moon always turns the same face to the Earth, and, if we neglect the small real (not apparent) librations, its angular velocity about the C′-axis is therefore constant and equal to n. Moreover, it is well known that its equator and the ecliptic intersect in a line whose longitude is w_3; call this point on the celestial sphere Ω. The mean angular distance of the A′-axis from Ω is therefore $w_1 - w_3$. Hence, from the right-angled spherical triangles having each a side, one on the ecliptic and the other on the equator, and a common hypotenuse ΩM,

$$\cos\delta' \cos(\alpha' + w_1 - w_3) = \cos U \cos(V - w_3).$$

If we neglect δ', U', this gives $\alpha' = V - w_1$. Put $\alpha' = V - w_1 + \delta\alpha'$; then $\delta\alpha'$ depends on squares of δ', U, and $\cos 2\alpha' = \cos 2(V - w_1) - 2\delta\alpha' \sin(V - w_1)$. As we shall neglect quantities of an order higher than the second with respect to the eccentricities and inclination, and also the inclination multiplied by m^2, we can neglect the second term of this last expression. Also if $\gamma = \sin\frac{1}{2}i$, and if $-i_1$ be the inclination of the lunar equator to the ecliptic (it being well known that the ecliptic lies between the mean lunar orbit and the lunar equator), we have with sufficient accuracy

$$\sin\delta' = \sin(i + i_1)\sin(V - w_3) = \sin(i + i_1)\sin(w_1 - w_3).$$

Substituting, we have, for the disturbing function

$$R = (E + M)\frac{a^2\mu'}{r^3}\left[\tfrac{1}{3} - \sin^2(i + i_1)\sin^2(w_1 - w_3) + (\mu''/\mu')\cos 2(V - w_1)\{1 - \sin^2(i + i_1)\sin^2(w_1 - w_3)\}\right],$$

where

$$a^2\mu' = \frac{3}{2M}\left(C' - \frac{A' + B'}{2}\right), \quad a^2\mu'' = \frac{3}{4M}(B' - A').$$

377. *Form for computation.*—The principal periodic terms have all short periods, and we need only consider the constant parts which give small constant additions to b_2, b_3. Now, if we retain quantities of the orders previously noted, that is, those parts which are of the second order with respect to e, γ, the portions depending on e^2 will alone affect b_2, and those depending on γ^2 will alone affect b_3. Hence for the former we can put

$$R=(E+M)\frac{a^2}{\rho^3}[\tfrac{1}{3}\mu'+\mu''\cos 2(V-w_1)];$$

and for the latter,

$$R=(E+M)\frac{a^2}{r^3}[-\tfrac{1}{2}\mu'-\tfrac{1}{2}\mu'']\sin^2(i+i_1),$$

in which terms of the order γ^2m^2 have been neglected.

Let ρ_e be the coefficient of e^2 in a^3/ρ^3, ρ_c that in $a^3\rho^{-3}\cos^2(V-w_1)$, or in $a^3\rho^{-5}u^2\zeta^{-2}$. Then, referring to the disturbing function for the figure of the Earth, and remembering the formulæ for obtaining δb_2, δb_3 (§ 270, Chap. X.), we see that the values of δb_2, δb_3 for the figures of the Moon and Earth are respectively in the ratios

$$\mu'\rho_e+3\mu''\rho_c : \mu(1-\tfrac{3}{2}\sin^2\epsilon_1)\rho_e \quad \text{and} \quad (\mu'+\mu'')\frac{d}{di}\sin^2(i+i_1) : \mu(1-\tfrac{3}{2}\sin^2\epsilon_1)\frac{d}{di}\sin^2 i.$$

378. *Numerical results.*—From the results in Sect. (v), Chap. X., and in the first section of this chapter we have

$$\rho_e=+\cdot386,\quad \rho_c=-\cdot678,\quad \mu=+[\bar{7}\cdot6658],\quad 1-\tfrac{3}{2}\sin^2\epsilon_1=+\cdot7623,\quad \delta b_2=+6''\cdot41,\quad \delta b_3=-6''\cdot00,$$

the last two being the values for the figure of the Earth. Hence, for the figure of the Moon, the annual mean motions in seconds of arc are

$$\delta b_2=8''\cdot41\frac{\mu'}{\mu}-44''\cdot3\frac{\mu''}{\mu},\qquad \delta b_3=-7''\cdot87\frac{\mu'+\mu''}{\mu}\frac{\sin 2(i+i_1)}{\sin 2i}.$$

I shall now assume that the Earth and Moon are of similar constitution, so that $C'/M : C/E$ in that ratio of the squares of their diameters, that is, as $(\cdot273)^2 : 1$. I also take

$$1-\frac{A+B}{2C}=\cdot00328,\quad i=5^\circ\cdot1,\quad i_1=1^\circ\cdot5\ (\S\,379).$$

Hence

$$\delta b_2=191''\left(1-\frac{A'+B'}{2C'}\right)-503''\left(\frac{B'-A'}{C'}\right),\qquad \delta b_3=-230''\left(1-\frac{A'}{C'}\right).$$

379. *Adopted values for the mechanical ellipticities.*—The results of Dr. F. Hayn,* for the lunar librations, give

$$i_1=1^\circ\cdot32',\quad B'-A'=+\cdot000157C',\quad C'-A'=+\cdot000629C'.$$

If we accept these values, we obtain for the annual mean motions

$$\delta b_2=+0''\cdot03,\quad \delta b_3=-0''\cdot14.$$

In order to obtain these quantities accurately to $0''\cdot01$, it is necessary to know the two mechanical ellipticities within 5 per cent. of their true values.

* *Abh. der Math.-Phys. Kl. der K.-Sächs. Gess. der Wiss.*, vol. xxx. (1907) p. 69.

CHAPTER XIV.

THE REMAINING PERTURBATIONS.

Section (i). *Corrections due to the Masses of the Earth and Moon.*

380. *Correction due to the substitution of m' instead of $m'+E+M$ for $n'^2a'^3$.* This is noted in § 4 (a), Chap. I. It amounts to diminishing the disturbing function due to the Sun by the factor $1-(E+M)/m'$. The method of Sect. (iii), Chap. XII., might be used, but it turns out to be troublesome. It is more simple to use the ordinary method for the indirect inequalities by putting $\delta\rho'=(E+M)/3m'$, $\delta V'=0$ for periodic terms, and $R_0=-(E+M)F_0/m'$ for the constant term, where F_0 is the constant part of the disturbing function due to the Sun.

381. *Results.*—I find, as in an earlier paper,* for the annual mean motions,

$$\delta b_2=-0''\cdot 69\,,\quad \delta b_3=+0''\cdot 19\,,$$

the constant changes of n, e, γ being insensible in the coefficients of the periodic terms.

The periodic changes are:

$$\delta V=-0''\cdot 007\sin 2\mathrm{D}-0''\cdot 020\sin(2\mathrm{D}-l)+0''\cdot 003\sin l'+0''\cdot 001\sin(l+l'-2\mathrm{D})\,.$$

382. *Corrections noted in § 2, § 4 (c) of Chap. I.*—The former gives $\delta b_2=-0''\cdot 01$, and the latter $\delta b_2=+0''\cdot 02$, $\delta b_3=-0''\cdot 01$.

Section (ii). *The Terms of the Second and Higher Orders.*

383. *Sources of the terms.*—In the four last chapters we have computed the perturbations due to various causes, on the assumption that certain factors which multiplied the disturbing functions were so small that their squares could be neglected. It remains to examine with some care to what extent this assumption is justifiable, and to correct the expressions in the cases where it is not true.

Let $R(r', V', z')$ be the disturbing function due to solar action, R_P that due to a planet, R_E that due to the figures of the Earth and Moon, and R_e that due to the motion of the ecliptic. Then if $\delta^2 r'$, $\delta^2 V'$, $\delta z'$ be the terms of the second

* *Monthly Notices*, vol. lvii. p. 567.

order in the motion of the Sun, the complete disturbing function for actions from all sources, except that in the main problem, is

$$\delta R = R(r'+\delta r'+\delta^2 r',\ V'+\delta V+\delta^2 V',\ z'+\delta z') - R(r',\ V,\ 0) + \Sigma R_P + R_E + R_r \quad . \quad . \quad (1).$$

We have previously neglected quantities of the order $(\delta R)^2$, and have used elliptic expansions for the coordinates of the Sun and planets in the last three terms.

384. It is convenient to divide the perturbations of the second order into classes according to their nature or the sources from which they arise. I denote by δ a perturbation of the first order, and by δ^2 one of the second order. The several portions of $\delta^2 R$ to be considered are as follows:—

(A) Terms due to the substitution of $c_i+\delta c_i$, $w_i+\delta w_i$ instead of c_i, w_i in the right-hand members of the equations of variations.

(B) $$\delta^2 r' . \frac{\partial R}{\partial r'} + \delta^2 V' . \frac{\partial R}{\partial V'} + \delta z' . \frac{\partial R}{\partial z'} .$$

(C) $$\tfrac{1}{2}\left(\delta r' . \frac{\partial}{\partial r'} + \delta V' . \frac{\partial}{\partial V'} + z' . \frac{\partial}{\partial z'}\right)^2 R.$$

(D) Additions due to periodic perturbations of the solar and planetary coordinates in ΣR_P.

(E) Changes in δR due to secular or quasi-secular variations of quantities which have been treated as constants.

(F) Changes due to the secular variations of the solar and planetary arguments.

(G) Third-order terms due to large second-order terms in the solar and planetary coordinates.

(H) Secular variations of the second order in general.

It is obvious, in the first place, that the only possibilities we have to consider are secular terms in w_1, w_2, w_3, and terms whose primaries are of very long period. In the second place, it is to be remembered that the variations of the elements contain terms of two kinds—those multiplied by the period of the primary, and those multiplied by the *square* of the period; and that the latter, for terms of very long period, are large compared with the former. Hence, except in the cases of arguments independent of the w_i (in which case the latter terms will be shown to disappear), it is sufficient to discuss only these latter terms. Finally, the greatest effect of these terms on the coordinates occurs through the change in w_1, so that it is generally sufficient to discuss $\delta^2 w_1$.

The various classes are considered in the following paragraphs.

385. (A) The canonical equations of Sect. (i), Chap. X., will be used and developed in a general manner for the second-order terms. The chief results obtained under this heading, namely, a proof that such terms are insensible, practically consists in showing that a given argument arises in two ways, and that, whenever the two

parts might be separately sensible, they are opposite in sign and nearly equal in magnitude.

Let $\delta^2 c_i$, $\delta^2 w_i$ be the additions to c_i, w_i due to the substitution of $c_i+\delta c_i$, $w_i+\delta w_i$ instead of c_i, w_i in the right-hand members. Then the equations for the determination of these additions are

$$\frac{d}{dt}\delta^2 c_i = \delta\frac{\partial R}{\partial w_i}, \quad \frac{d}{dt}\delta^2 w_i = -\delta\frac{\partial R}{\partial c_i} + \Sigma_k\left(\frac{db_i}{dc_k}\delta^2 c_k\right) + \tfrac{1}{2}\left(\Sigma_k \delta c_k \frac{d}{dc_k}\right)^2 b_i \quad . \quad . \quad . \quad (2).$$

Let the first-order variations δc_i, δw_i be due to a term * in R

$$\lambda \cos\phi = \lambda\cos(pt+\alpha) = \lambda\cos(j_1 w_1 + j_2 w_2 + j_3 w_3 + \psi),$$

where ψ is independent of the w_i. Then

$$\delta c_i = j_i \frac{\lambda}{p}\cos\phi, \quad \delta w_i = \left(\frac{\lambda}{p^2}\frac{dp}{dc_i} - \frac{1}{p}\frac{d\lambda}{dc_i}\right)\sin\phi \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Let any other term of R be

$$\lambda'\cos\phi' = \lambda'\cos(p't+\alpha') = \lambda'\cos(j_1' w_1 + j_2' w_2 + j_3' w_3 + \psi'),$$

where ψ' is independent of the w_i. Let us consider these two terms alone; then the terms in $\delta^2 c_i$, $\delta^2 w_i$ will have the arguments $\phi\pm\phi'$.

Put

$$j_1\frac{\partial}{\partial c_1} + j_2\frac{\partial}{\partial c_2} + j_3\frac{\partial}{\partial c_3} = \frac{\partial}{\partial c}, \quad j_1'\frac{\partial}{\partial c_1} + j_2'\frac{\partial}{\partial c_2} + j_3'\frac{\partial}{\partial c_3} = \frac{\partial}{\partial c'} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Then

$$\delta\frac{\partial R}{\partial w_i} = \Sigma_k\left(\frac{\partial^2 R}{\partial w_i \partial c_k}\delta c_k + \frac{\partial^2 R}{\partial w_i \partial w_k}\delta w_k\right)$$

$$= -j_i'\frac{\lambda}{p}\cdot\frac{d\lambda'}{dc}\sin\phi'\cos\phi - j_i'\left(\frac{\lambda\lambda'}{p^2}\cdot\frac{dp}{dc'} - \frac{\lambda'}{p}\cdot\frac{d\lambda}{dc'}\right)\cos\phi'\sin\phi.$$

To this must be added the term arising by making δw_i, δc_i depend on ϕ', and the derivatives of R on ϕ, that is, by changing the accents.

We thus get

$$\frac{d}{dt}\delta^2 c_i = \left[\lambda\frac{d\lambda'}{dc}\left(\frac{j_i}{p'} - \frac{j_i'}{p}\right) - j_i\frac{\lambda\lambda'}{p'^2}\cdot\frac{dp'}{dc}\right]\sin\phi'\cos\phi + \left\{\begin{matrix}\text{similar term with}\\ \text{accents changed}\end{matrix}\right\},$$

and therefore, since $db_i/dc_k = db_k/dc_i$,

$$\frac{d}{dt}\sum_k \frac{db_i}{dc_k}\delta^2 c_k = \tfrac{1}{2}\left[\left(\lambda'\frac{d\lambda}{dc'} \mp \lambda\frac{d\lambda'}{dc}\right)\left(\frac{1}{p}\cdot\frac{dp'}{dc_i} - \frac{1}{p'}\cdot\frac{dp}{dc_i}\right) - \lambda\lambda'\left(\frac{1}{p^2}\cdot\frac{dp}{dc'}\cdot\frac{dp'}{dc_i} \pm \frac{1}{p'^2}\cdot\frac{dp'}{dc}\cdot\frac{dp}{dc_i}\right)\right]\sin(\phi\pm\phi').$$

Denote the right-hand member of this equation by $F_i\sin(\phi+\phi') + F_i'\sin(\phi-\phi')$.

* The symbols j_1, j_2, j_3 are the same as the i_1, i_2, i_3 of Chap. X.; the j_1, j_2, j_3 of that chapter are not needed here.

Again,

$$\delta\frac{\partial R}{\partial c_i}=\sum_k\left(\frac{\partial^2 R}{\partial c_i\partial c_k}\delta c_k+\frac{\partial^2 R}{\partial c_i\partial w_k}\delta w_k\right)$$

$$=\frac{d^2\lambda'}{dcdc_i}\,\frac{\lambda}{p}\cos\phi\cos\phi'-\frac{d\lambda'}{dc_i}\left(\frac{\lambda}{p^2}\cdot\frac{dp}{dc'}-\frac{1}{p}\cdot\frac{d\lambda}{dc'}\right)\sin\phi\sin\phi'+\text{two similar terms}$$

$$=\tfrac{1}{2}\left[\frac{\lambda}{p}\cdot\frac{d^2\lambda'}{dcdc_i}+\frac{\lambda'}{p'}\cdot\frac{d^2\lambda}{dc'dc_i}\pm\left\{\frac{d\lambda'}{dc_i}\left(\frac{\lambda}{p^2}\cdot\frac{dp}{dc'}-\frac{1}{p}\cdot\frac{d\lambda}{dc'}\right)+\begin{matrix}\text{similar}\\ \text{term}\end{matrix}\right\}\right]\cos(\phi\pm\phi'),$$

which is denoted by $G_i\cos(\phi+\phi')+G_i'\cos(\phi-\phi')$.

Finally,

$$\tfrac{1}{2}\left(\sum_k\delta c_k\frac{d}{dc_k}\right)^2 b_i=\frac{\lambda\lambda'}{pp'}\left(\sum_k j_k'\frac{d}{dc_k}\right)\left(\sum_k j_k\frac{d}{dc_k}\right)b_i=\frac{\lambda\lambda'}{pp'}\,\frac{d^2p'}{dcdc_i}\cos\phi\cos\phi'=H_i\cos(\phi\pm\phi').$$

Substituting these results in the second three of equations (2), we obtain

$$\delta^2 w_i=\int(H_i-G_i)\cos(\phi+\phi')dt+\iint F_i\sin(\phi+\phi')dt+\text{two similar terms}$$

$$=\left(\frac{H_i-G_i}{p+p'}-\frac{F_i}{(p+p')^2}\right)\sin(\phi+\phi')+\left(\frac{H_i-G_i'}{p-p'}-\frac{F_i'}{(p-p')^2}\right)\sin(\phi-\phi')\quad . \quad . \qquad (5).$$

Since we have only to search for terms of very long period, the most important terms are those which acquire the squares of the small divisors $p\pm p'$, and therefore the coefficients F, F' are of chief importance.

386. (α) *If p, p' are independent of the w_i*, we have $\delta^2 c_i=0$, $\delta^2 w_i=0$.

387. (β) *If $j_i=j_i'$ and $p\doteq p'$*, the argument $\phi-\phi'$ is of long period. In this case we have

$$\frac{dp'}{dc_i}=\frac{dp}{dc_i},\quad \frac{dp}{dc}=\frac{dp'}{dc}=\frac{dp}{dc'}=\frac{dp'}{dc}.$$

Hence F' contains the factor $p-p'$, and $\delta^2 w_i$ only acquires the first power of the small divisor $p-p'$.

In no case, therefore, can the small divisor appear in its second power for terms independent of the lunar angles.

388. (γ) *If $j_i=j_i'$ and $p\doteq -p'$*, the argument $\phi+\phi'$ is of long period. Then the possible combinations from the first-order terms show that one of two things must happen: either p, p' are not small, in which case λ, λ' are so minute that there is no possibility of a sensible coefficient; or p, p' are themselves small. In the latter case, the second term of F_i is the principal one, and it gives for the coefficient, since $p\doteq -p'$,

$$-\tfrac{1}{2}\frac{\lambda\lambda'}{(p+p')^2}\left(\frac{1}{p^2}+\frac{1}{p'^2}\right)\left(\frac{dp}{dc}\right)\frac{dp}{dc_i}\doteq-\left(\frac{\lambda}{p^2}\cdot\frac{dp}{dc}\right)\left(\frac{\lambda'}{p'^2}\cdot\frac{dp'}{dc_i}\right)\frac{p^2}{(p+p')^2}.$$

This is of the order $\delta w_i\,.\,\delta' w_i\,.\,p^2/(p+p')^2$, where δw_i, $\delta' w_i$ are the first-order terms with the arguments ϕ, ϕ'. If C, C' are the coefficients of δw_i, $\delta w_i'$ expressed in seconds of arc, the order of the new coefficient, also expressed in seconds of arc, is

$$\frac{CC'}{206265}\left(\frac{p}{p+p'}\right)^2.$$

389. (δ) *If* $j_1 = j_1'$ *and* $p \doteq p'$, the terms in F_1' which have the divisor $(p-p')^2$ have also as a factor one of the derivatives of b_2 or b_3 to c_2 or c_3, that is, the factor m^2. If $p \doteq -p'$, the argument in (γ) still holds for the principal parts of $\delta^2 w_1$, the most important variation.

390. (ϵ) *If* $j_i' = 0$, *and* $p \doteq p'$, then $-F_i'/(p-p')^2$ becomes

$$\frac{\lambda}{p'} \cdot \frac{d\lambda'}{dc} \cdot \frac{dp}{dc_i} \cdot \frac{1}{(p-p')^2} = (j_1 \delta' w_1 + j_2 \delta' w_2 + j_3 \delta' w_3)\left(-\frac{\lambda}{p^2} \cdot \frac{dp}{dc_i}\right)\left(\frac{p}{p-p'}\right)^2,$$

$H_i = 0$, and

$$-\frac{G_i'}{p-p'} = -\tfrac{1}{2}\left[\frac{\lambda}{p} \cdot \frac{d^2\lambda'}{dc\,dc_i} + \frac{d\lambda}{dc_i} \cdot \frac{1}{p'} \cdot \frac{d\lambda'}{dc}\right]\frac{1}{p-p'} \doteq -\tfrac{1}{2}\frac{p}{p-p'} \cdot \frac{d}{dc_i}\left(\frac{\lambda}{p'^2} \cdot \frac{d\lambda'}{dc}\right).$$

The principal term is the former of these two.

Other cases are treated in like manner. The net result is that terms independent of the lunar arguments only acquire the small divisor $p-p'$, while those containing the lunar arguments are at most of order $(\delta w_i)(\delta' w_i)p^2/(p-p'^2)$. A somewhat closer examination is only necessary when the latter class of terms cannot be neglected at this first test. No sensible terms have been found.

391. (ζ) *For secular terms*, we have $\delta c_i = 0$, $\partial R/\partial w_i = 0$, and therefore $\delta^2 c_i = 0$, $\delta^2 w_i = 0$. However, certain changes have been made in the arbitrary constants giving changes $\delta_0 c_i$. But these only produce constant changes in the $d\delta^2 w_i/dt$, and therefore further changes in the arbitraries; the latter are insensible.

But secular terms may arise from the combination of two terms of the first order of the form $(a+bt)\sin\phi$, $(a'+b't)\sin\phi'$, where ϕ, ϕ' are either equal or where their periods differ by so small an amount that $\phi-\phi'$ may be treated as secular, and a, a', b, b' are constants. Owing to the minuteness of b, b', such terms are entirely insensible.

392. (B) These terms are treated in exactly the same way as the indirect terms of the first order. They are computed in § 402 below. There are no sensible portions depending on $\delta z'$.

393. (C) These terms are of the order of the indirect disturbing function of Chap. XII. multiplied by $3\delta\rho'$ or $\delta V'$. If p be the mean motion of a term obtained in that chapter, and p' that of a term in $\delta\rho'$ or $\delta V'$, the worst case is easily seen to be of the order

$$\frac{3 \cdot 10}{206265} \cdot \frac{p}{p-p'} = \frac{1}{7000} \cdot \frac{p}{p-p'} \text{ seconds of arc},$$

for terms independent of the lunar angles, since the largest term in $\delta\rho'$ or $\delta V'$ is less than $10''$, and there is no term due to indirect action having a coefficient so great as $2''$.

For terms dependent on the lunar angles we obtain the order

$$\frac{1}{7000}\left(\frac{p}{p-p'}\right)^2 \text{ seconds},$$

and a brief examination is necessary. It is easy to see that, if w_1 is present, $\delta\rho'$, $\delta V'$ must have periods comparable with the month, and must, therefore, have coefficients

too minute to produce sensible effects. When w_1 is not present, there are only one or two terms to consider (the most probable of which arise from the action of Jupiter); these were examined and were found to be quite insensible.

394. (D) These inequalities cause additions

$$\delta^2 R = \left[\delta\rho'(-3-I) + \delta\rho''\cdot I + \delta(V'-V'')\left(\frac{d}{d\epsilon'} + \frac{d}{d\varpi'}\right)\right]\Sigma R_P \quad \text{for inner planets}, \quad . \quad . \quad (6),$$

$$\delta^2 R = \left[\delta\rho \cdot I + \delta\rho''(-3-I) + \delta(V'-V'')\left(\frac{d}{d\epsilon'} + \frac{d}{d\varpi'}\right)\right]\Sigma R_P \quad \text{for outer planets}, \quad . \quad . \quad (7),$$

where $\delta\rho''$, $\delta V''$ refer, in any term, to the same planet as that to which the R_P for that term belongs; R_P is confined to that part independent of a/a', and I is defined in § 296, Chap. XI.

To obtain the order of magnitude of the coefficient it will be sufficient to consider terms for which I or the derivatives with respect to ϵ', ϖ' produce a factor not greater than 20. Also, since $\delta\rho'$, $\delta\rho''$ are of the same or of higher order of small quantities than $\delta V'$, $\delta V''$, we use the latter. Proceeding as in case (C), we obtain for the maximum order of $\delta^2 w_1$

$$20\frac{\delta V\cdot\delta(V'-V'')}{206265}\left(\frac{p}{p-p'}\right)^2 \doteq \frac{\delta V\cdot\delta(V'-V'')}{10000}\left(\frac{p}{p-p'}\right)^2,$$

in which every term, as well as the result, is expressed in seconds of arc.

For inequalities independent of the Moon angles, the factor $p/(p-p')$ only occurs in its first power, and we use $\delta\rho'$ or $\delta\rho''$ in the place of $\delta(V'-V'')$.

This method provides for terms which arise with a sensible first-order term, that is, with a term in δV already obtained. But there may be other combinations due to terms in δV previously neglected. Since $\delta(V'-V'')$ must be very minute for terms of short period relative to the year, we need only consider terms of long period in δV relative to the month, and from such terms those containing multiples of w_1 higher than the first can be certainly excluded. A detailed consideration (which will be omitted) of the manner in which the various quantities enter into δV gives a maximum value for $\delta^2 w_1$, in the case of Venus, of about $20\delta(V'-V'')/s^2$, where s is the number of seconds in the daily motion of the angles $\phi \pm \phi'$, and $\delta V'$, $\delta V''$ express the number of seconds of arc in the coefficients of the terms in V', V'' under consideration.

395. First let us exclude from $\delta V''$ all perturbations but those produced by the Earth. Then the maximum value of $\delta(V'-V'')$ is $10''$, and if we neglect coefficients in $\delta^2 w_1$ less than $0''\cdot 1$, we obtain a maximum value for s of $44''$. Similar considerations for all the planets give numbers exhibited in the following table, which shows the maximum value of s for different combinations of $\delta V'$ with R_P or R_E. In the principal diagonal, where the suffixes are the same, $\delta(V'-V'')$ must be understood; the suffixes denote the sources of the terms. Values of s less than $1''$ are put down as zeros. The last row refers to the ellipticity terms; R_e is too small to give anything sensible.

Maxima of s *for a Coefficient in* δV *of* $0''\cdot 1$.

	$\delta V_J'$	$\delta V_V'$	$\delta V_M'$	$\delta V_S'$	$\delta V_Q'$
R_J	44	44	24	24	2
R_V	44	60	24	24	2
R_M	7	7	7	4	0
R_S	14	14	7	7	0
R_Q	7	7	4	4	2
R_E	44	44	7	14	7

Since the coefficient in $\delta^2 V$ varies inversely as the square of s, this table shows that for a coefficient of $1''$ in $\delta^2 V$ the value of s must be less than $6''$, or the period greater than 600 years.

The numbers in this table are only rough approximations; but even if they ought to have been twice as great, it would simply have meant that the corresponding coefficient in $\delta^2 V$ should be taken as $0''\cdot 4$ instead of $0''\cdot 1$. It will in any case retain any sensible coefficients. In fact, for periods of over 100 years or so, the minimum sensible coefficient will certainly be less than $0''\cdot 1 t_c$, where t_c is the number of centuries in the period.

The various combinations have been considered in the same manner as that employed in Sect. (iii), Chap. XI. There appeared to be only one which might give a sensible coefficient, namely, the combination

$$(l + 16\text{T} - 18\text{V}) - (13\text{T} - 8\text{V}) = l + 3\text{T} - 10\text{V},$$

which has a period of 1900 years. This is therefore a second approximation to a term due to direct action and given in Chap. XI.

There are no sensible combinations of terms independent of the lunar angles.

396. In the above, all terms in $\delta V''$, $\delta\rho''$ which arise from planets other than the Earth have been omitted, but for these omitted terms the mutual perturbations in the lunar disturbing function cancel one another very nearly. For example, the great inequalities in Jupiter and Saturn with argument $5\text{S} - 2\text{J}$ appear in both R_S and R_J. Now, LEVERRIER has shown * that the effect of such terms on the motion of the Earth may be neglected provided we take J, S to represent the mean longitude, as affected by the great inequality instead of the mean longitude alone. But, for such terms, the lunar disturbing function for the direct action only differs from the Earth's disturbing function by certain constant factors and by certain operators $k_0 + k_1 a'd/da' + k_2(a'd/da')^2$, k_0, k_1, k_2, . . being certain constants independent of the disturbing functions (Sect. (ii), Chap. XI.). The terms produced by k_0 nearly cancel one another, while Jupiter and Saturn are so far away that the other terms are very small. A rough calculation shows that terms from this source are quite insensible.

* *Ann. Obs. Paris*, vol. ii.

Hence, the terms of this nature are sufficiently accounted for by adding the planetary perturbations to the mean motions of the arguments of the planets.

397. (E) For these secular terms, the investigation of Sect. (iii), Chap. XII., shows that it is sufficient to insert the disturbed values in the final results. The only term which can be affected is the great inequality due to Venus. This depends chiefly on γ''^2 and $\delta\gamma''/\gamma'' = t_c/3400$, where t_c is the number of centuries from 1850·0. The maximum change is therefore less than 0″·01 per century and is quite inappreciable.

398. (F) These motions should properly be inserted in the arguments before division by s or s^2, and this is sufficient to account for them. There are only two terms which can be sensibly affected, those with arguments $l + 16T - 18V$, and $l + 3T - 10V$. The term with argument $2\psi + 2\varpi'$, period 10,000 years, due to the figure of the Earth, has a coefficient less than 0″·01 (according to Dr Hill,* 0″·0025).

399. (G) There are a few second-order terms in the motions of the planets which, when inserted in the disturbing function for the direct action, might produce sensible third-order terms in the Moon's motion. A list of these was made and the possible combinations considered, but nothing sensible was found.

400. (H) *Secular variations.*—It is a fact well known in the planetary theory that the secular variations of the planetary elements do not produce secular variations in the function which is the inverse distance between the planets. But the disturbing function which has been used for the direct action depends mainly on derivatives of the planetary disturbing function with respect to the Earth's distance and mean longitude. Hence the secular motions of the planetary perihelia and nodes can produce no secular changes in the Moon's motions through the direct action. In the same way the part arising through $\delta a'$ in the indirect action is insensible.

We have then the ellipticity terms to consider. Here we have taken ϵ_1, the inclination of the ecliptic (which was considered as fixed) to the equator, to be constant. When the motion is referred to the moving plane of reference, it might be thought that this would introduce a secular term of the second order. But the principal part of the motion of the ecliptic only produces a *periodic* term of period equal to that of the Moon's node, and the principal perturbation produced by the figure of the Earth is also a periodic term whose period is that of the Moon's node plus the precession. The term of the second order which might be sensible is therefore one having a period which is the difference between these periods, that is, a term independent of the lunar angles and of period equal to that of the precession; it is therefore quasi-secular. But it is easier to treat it by the method of § 385, and then the theorem that the first power of its period will be the only multiplier tending to make the coefficient large comes into play. If we expand the term in the form $a + bt_c + ct_c^2 + \ldots$, the portion $a + bt_c$ is only a slight alteration to the mean longitude (an observed quantity), and the secular part ct_c^2 will therefore have the first power of the period as a *divisor*, and consequently may be expected to be very small.

* *Amer. Eph. Papers*, vol. iii. p. 342; or *Coll. Works*, vol. ii. p. 318.

The period is somewhat altered owing to the motion of τ, but the argument is unaffected, since τ is only present with w_3.

I have computed this term in the following section, and have found $0''\cdot 15$ for the coefficient of the periodic term, and consequently, for the secular acceleration during historic times, $0''\cdot 0001 t_c^2$, a quantity quite insensible.

The theorem that periodic terms of the first and second orders independent of the lunar angles can never receive multipliers higher than the first power of the long period practically enables one to reject any possibilities of secular or quasi-secular terms arising from perturbations of the second order, whatever may be the source.

401. *Summary.*—We have to find :

Case (B). The indirect effect of the solar terms whose arguments are $4M - 7T + 3V$, $3J - 8M + 4T$. In longitude these are, according to NEWCOMB,*

$$\delta^2 V' = +0''\cdot 266 \sin (332^\circ\cdot 3 + 119^\circ\cdot 0 t_c) + 6''\cdot 40 \sin (221^\circ\cdot 1 + 20^\circ\cdot 2 t_c),$$

where t_c is the number of centuries from 1850·0.

Case (D). The term of argument $l + 3T - 10V$ due to substitution of the periodic variations of the coordinates of the Earth and Venus of period $13T - 8V$ in the disturbing function for the direct action.

Case (F). The insertion of the motions of the perihelia and nodes in the arguments $l + 16T - 18V$, $l + 3T - 10V$, with the resulting changes in the divisors s^2 and the coefficients of the terms.

Section (iii). *Calculation of the Terms.*

402. Case (B). We require $\delta^2\rho'$, and we only know $\delta^2 V'$. Now, $\delta^2\rho'$ depends mainly on $\delta^2 a'/a'$, and for terms of very long period we have

$$\delta^2 V' = \delta^2 T = -\tfrac{3}{2}\int \frac{\delta^2 a'}{a'} n' dt,$$

so that, approximately,

$$\delta^2\rho' = \frac{\delta^2 a'}{a'} = -\tfrac{2}{3}\frac{1}{n'}\frac{d}{dt}(\delta^2 V').$$

But the methods of Sect. (i), Chap. X., and of Sect. (i), Chap. XII., give

$$\delta^2 T = \frac{1}{a^2\beta}\int \frac{d}{dn}\delta^2 R dt = -\tfrac{3}{4}\frac{n'^2}{a^2\beta}\int \delta^2\rho' \frac{d}{dn}\{(M_1 + \tfrac{3}{2}M_2)_0 a^2\} dt$$

$$= \tfrac{1}{2}\frac{n'}{n}\cdot\frac{1}{\beta a^2}\cdot n\frac{d}{dn}\{(M_1 + \tfrac{3}{2}M_2)_0 a^2\}\delta^2 V' = -[\bar{1}\cdot 1193]\delta^2 V',$$

approximately ; here the suffix 0 denotes the values of M_1, M_2 from Sect. (v), Chap. X., corresponding to $\theta = 0$.

This formula, similar to that obtained for first-order terms by RADAU,† who, however, neglects M_2, gives for the two terms mentioned in the previous section

$$\delta^2 w_1 = +0''\cdot 04 \sin (152^\circ + 119^\circ\cdot 0 t_c) + 0''\cdot 84 \sin (41^\circ\cdot 1 + 20^\circ\cdot 2 t_c).$$

* *Tables of the Sun*, p. 19.

† *L.c.*, § 279, p. 39.

403. Case (D). The term of argument $l+16T-18V+151^\circ\cdot0=\phi$ arises from a term of this argument in R_P. The term of argument $l+3T-10V+\alpha^\circ=\phi-\phi'$ arises chiefly from a term in $\delta^2 R$,

$$\left(\frac{dR_V}{d\epsilon'}+\frac{dR_V}{d\varpi'}\right)\delta(T-V),$$

where δT is the term of argument ϕ' in the Earth's mean longitude, and δV a term of the same argument in the mean longitude of Venus.

From Newcomb's computations* $\delta V=-(1\cdot92/1\cdot44)\delta T$, so that

$$\delta T-\delta V=+\frac{3\cdot36}{1\cdot44}1''\cdot89\sin(13T-8V+134^\circ)=+4''\cdot41\sin(13T-8V+134^\circ).$$

To deduce the new term from that with argument ϕ which has a coefficient in δV of $14''\cdot5$, we must multiply by -16 to account for $\partial/\partial\epsilon'+\partial/\partial\varpi'$, by $(13\cdot01/1\cdot85)^2$ for the change in the value of s^2, by $\delta T-\delta V$, by $\frac{1}{2}$ for the change of the product of two sines into the cosine of the difference of ϕ, ϕ', and divide by 206265 for the reduction to seconds. This gives

$$\begin{aligned}\delta^2 V &= (-16)(14\cdot5)\left(\frac{13\cdot01}{1\cdot85}\right)^2(\tfrac{1}{2})(4\cdot41)\frac{1}{206265}\sin(l+3T-10V+17^\circ)\\ &= -0''\cdot12\sin(l+3T-10V+17^\circ).\end{aligned}$$

This reduces the term of this period (see § 316) to $+0''\cdot23\sin(l+3T-10V+21^\circ)$.

404. Case (F). The principal part of the term with argument $l+16T-18V+151^\circ\cdot0$ is obtained from $l+16T-18V+2h''$, where h'' is the longitude of the node of Venus. This node has a daily motion of $-0''\cdot05$, so that the daily motion of the argument is $-13''\cdot11$ instead of $-13''\cdot01$. The coefficient $(14''\cdot49)$ must therefore be diminished in the ratio of the squares of these two numbers, that is, by $0''\cdot22$. This is the only sensible effect that terms of the second order have on the great Venus inequality.

The coefficient of the other term depends chiefly on $l+3T-10V+\varpi'+6h''$. The daily motion of the two latter terms of this argument, which were the parts previously neglected, is $-0''\cdot26$, and of the former three, $+1''\cdot85$. Hence the coefficient must be increased in the ratio $(1\cdot85/1\cdot59)^2$. It was found in case (D) to be $0''\cdot23$; its final value is therefore $+0''\cdot31$.

The motion of τ in the argument $w_3-\tau$ due to the motion of the ecliptic was neglected. It has a daily mean motion of $+0''\cdot09$, so that the coefficients must be diminished by about $1/1800$ of their value.

405. *A special term in the secular acceleration due to the motion of the ecliptic and the figure of the Earth.*—The arguments of the principal terms due to the motion of the ecliptic and the figure of the Earth are respectively $\phi=w_3+96^\circ\cdot2-0''\cdot09t$, $\phi=w_3+0''\cdot14t$, where the coefficients of t are the daily motions of τ and of the precession respectively. The argument $\phi-\phi'$ is therefore $-0''\cdot23t+96^\circ\cdot2$. This is quasi-secular, but it is more easily treated as periodic by the formulæ of case (A) of the

* *Am. Eph. Papers*, vol. iii. pp. 476, 488.

previous section. It is independent of the w_i, and therefore only receives the first power of the period as a large multiplier.

We have $j_1 = j_2 = j_1' = j_2' = 0$, $j_3 = j_3' = 1$, $p = b_3 - 0''\cdot 09$, $p' = b_3 + 0''\cdot 14$. Since b_3 has a daily motion of $-190''\cdot 8$, we can put $p = p' = b_3$, except when they occur in the combination $p - p'$.

Substituting in the formulæ of § 385, and remembering that we only need $\delta^2 w_1$,

$$\frac{-F_1'}{(p-p')^2} = \tfrac{1}{2} \cdot \frac{1}{p-p'}\left(\lambda' \frac{d\lambda}{dc_3} + \lambda \frac{d\lambda'}{dc_3}\right) \frac{1}{b_3^2} \frac{db_3}{dc_1} - \frac{\lambda\lambda'}{p-p'} \cdot \frac{1}{b_3^2} \cdot \frac{db_3}{dc_1} \cdot \frac{db_3}{dc_3}.$$

Now λ, λ' contain the factor k, and $c_3 = -2k^2 na^2$ approximately. Hence

$$\frac{-F_1'}{(p-p')^2} = \frac{\lambda\lambda'}{2b_3^2(p-p')} \frac{1}{c_3} \frac{db_3}{dc_1}\left[1 - \frac{2c_3}{b_3} \cdot \frac{db_3}{dc_3}\right].$$

The second term in the square bracket is of order k^2 compared with the first, and may therefore be neglected.

Treating G_1' in the same manner, and neglecting the terms factored by db_3/dc_3, which are of order k^2 compared with the others, we find, after some reductions,

$$\frac{G_1'}{p-p'} = \tfrac{1}{2} \cdot \frac{1}{c_3 b_3 (p-p')}\left(\lambda \frac{d\lambda'}{dc_1} + \lambda' \frac{d\lambda}{dc_1}\right) = \tfrac{1}{2} \cdot \frac{1}{c_3 b_3 (p-p')} \cdot \frac{d}{dc_1}(\lambda\lambda').$$

Also, H_1 is of order k^2 compared with these terms. Hence

$$\delta^2 w_1 = \frac{1}{2(p-p')} \cdot \frac{1}{b_3 c_3}\left[\frac{\lambda\lambda'}{b_3} \cdot \frac{db_3}{dc_1} + \frac{d}{dc_1}(\lambda\lambda')\right] \sin(\phi - \phi').$$

Now, by Sect. (ii), Chap. X., approximately,

$$\frac{d}{dc_1} = -\frac{1}{a^2\beta} \frac{d}{dn}, \quad \beta = \cdot 33, \quad \frac{db_3}{dn} = +\cdot 0037, \quad b_3 = -\cdot 0040.$$

Also, $\lambda = ka^2 n\mu$, $\lambda' = kn^2\mu'$, where μ, μ' are independent of the c_i for the terms of lowest order, so that $\lambda\lambda' = k^2 a^2 n^3 \mu\mu' \doteq c_3 n^2 \mu''$, where μ'' is defined like μ, μ'. Hence

$$\delta^2 w_1 = \tfrac{1}{2} \cdot \frac{p}{p-p'} \frac{\lambda\lambda'}{b_3^2 c_3 na^2}[3-6] \sin(\phi - \phi').$$

But, approximately,

$$-4na^2\gamma\delta\gamma = \delta c_3 = \frac{\lambda}{b_3}\cos\phi, \qquad -4na^2\gamma\delta'\gamma = \frac{\lambda'}{b_3}\cos\phi';$$

and $\delta\gamma = +0''\cdot 70 \cos\phi$ (§ 370), $\delta'\gamma = -4''\cdot 35 \cos\phi'$ (§ 374), $p = b_3 = -190''\cdot 8$, $p - p' = -0''\cdot 23$, $\phi - \phi' = 96^\circ\cdot 2 - 0''\cdot 23t$. Hence

$$\delta^2 w_1 = -12 \cdot \frac{190\cdot 8}{-\cdot 23} \cdot \frac{(\cdot 70)(4''\cdot 35)}{206265} \sin(96^\circ\cdot 2 - 0''\cdot 23t) = 0''\cdot 15 \sin(96^\circ\cdot 2 - 2^\circ\cdot 3t_c),$$

which is equivalent to a modern secular acceleration of $0''\cdot 0001 t_c^2$, and therefore entirely insensible.

406. It might still be thought that terms of the second order in the disturbing function of Sect. (iv), Chap. XII., will give rise to sensible secular terms. But θ_3 is of the third order, and therefore any terms which arise will either depend on the w_i or will be constants. It is also to be remembered that arguments in x, y containing odd

multiples of w_3, and arguments in z containing even multiples of w_3, only arise through the disturbing functions δR, $\delta^2 R$, and much the largest of the combinations with a long period and of the second order is that just computed.

Section (iv). *Perturbations with Unknown Constants.*

407. *The attractions of the minor planets.*—No one of these is large enough to produce any sensible effect on the motion of the Moon, but their aggregate mass may possibly be comparable with the mass of the Earth. The chief force would be that of a ring of matter of diameter between two and three times that of the Earth's orbit, and the principal effect would be constant additions to the mean motions of the perigee and node of the Moon's orbit. But on any supposition which would fit in with the small differences between theory and observation for the motions of the planets, these constant additions cannot exceed two or three hundredths of a second of arc in the annual mean motions of the Moon's node and perigee.

Periodic effects would be smaller, and the chief of them would have a period of one year, with a coefficient less than one-thousandth of a second of arc.

408. *Other matter in the solar system.*—There is undoubtedly a large number of meteoric bodies distributed through the solar system and revolving mainly round the Sun. The most reasonable supposition is that this matter may be considered as arranged in rings of varying density round the Sun as centre, so that the effect would be that of a thin plate with its centre at the Sun, and of density increasing towards the centre. If this density varied as the qth power of the distance from the Sun, where q is some negative number, the effect would again be to add to the apparent mass of the Sun, to add something to the mean motions of the perigee and node, and to produce additions to the known periodic terms. The effect on the Earth would be of a similar nature. No secular terms arise. Professor NEWCOMB has discussed such hypotheses.* In any case, the effect on the Moon can be neglected.

409. *The action of the tides.*—First, neglecting friction, the action of the lunar tide chiefly produces a standing wave with reference to the line joining the Earth and Moon. Its effect is therefore similar to that of the figure of the Moon, and there can be little doubt that it is quite negligible. The action of the solar tide must chiefly produce a term depending on the difference of longitudes of the Sun and Moon, and is similarly too small to be considered. The reaction of tidal friction produces a real secular retardation of the Moon's motion, as well as the apparent acceleration due to the slowing down of the Earth's rotation. The former is nearly equal to the latter, and the real retardation would be between two and three times the observed acceleration. There being no data on which to base any exact numerical estimates of either of these quantities, the secular acceleration will be considered as an observed quantity, the magnitude of the apparent value being not very different from that ($5''\cdot 8$) found from the attractions of the planets.

* *Astronomical Constants*, chap. vi.

CHAPTER XV.

THE FINAL EXPRESSION FOR THE MOON'S COORDINATES.

410. In this concluding chapter I gather together all the perturbations which have been found in detail in Chaps. X.–XIV., so that the expressions for the coordinates of the Moon in terms of the time are obtained by adding the results given below to those at the end of Chap. IX.

411. The values of the mean motions of the perigee and node are collected in the following scheme with the references.

Annual mean motion of the	*Perigee.*	*Node.*
Principal solar action (§ 195)	$+146426''\cdot92$	$-69672''\cdot04$
Mass of the Earth (§§ 381, 382)	$-\ \cdot68$	$+\ \cdot19$
Direct planetary action (§ 316)	$+\ 2\cdot69$	$-\ 1\cdot42$
Indirect ,, ,, (§ 370)	$-\ \cdot16$	$+\ \cdot05$
Figure of the Earth (§ 374)	$+\ 6\cdot41$	$-\ 6\cdot00$
,, ,, Moon (§ 379)	$+\ \cdot03$	$-\ \cdot14$
Final values (epoch 1850·0)	$+146435\cdot21$	$-69679\cdot36$

[The small differences from the values given in 1904 * are chiefly due to the somewhat doubtful parts depending on the figure of the Moon; these were neglected in the earlier paper. The value of the ellipticity of the Earth adopted here is that corresponding to the result marked (β) in the paper.]

412. *Notation.*—I now use w_1, w_2, w_3 to represent the mean longitude and the mean longitudes of the perigee and node with the motions just given; ϖ', the mean longitude of the perihelion of the Earth's orbit at epoch; Q, V, T, M, J, S, the mean longitudes of the planets. These will receive additions in § 413 below, denoted by the symbol δ. Thus

$$\mathrm{D}=w_1+\delta w_1-\mathrm{T}-\delta\mathrm{T}\,,\quad l=w_1+\delta w_1-w_2-\delta w_2\,,\quad \mathrm{F}=w_1+\delta w_1-w_3-\delta w_3\,,\quad l'=\mathrm{T}+\delta\mathrm{T}-\varpi'-\delta\varpi'.$$

The constants γ, e' also receive additions $\delta\gamma$, $\delta e'$, given below. The changes in the Moon's coordinates are accounted for if we multiply the terms containing the arguments $i\mathrm{F}$, il' by the variable factors

$$1+i\,\frac{\delta\gamma}{\gamma}\cdot\frac{1}{206265}\,,\qquad 1+i\,\frac{\delta e'}{e'}\,,$$

* *Monthly Notices*, vol. lxiv. p. 532.

respectively. All other variations of the constants present in the coefficients are either insensible or have been included in the expressions for the coordinates.

The number of centuries from 1850·0 is represented by t_c.

The arrangement of the tables is the same as that of Chaps. XI., XII.

All final coefficients below $0''.003$ have been dropped.

A star instead of a number in the last place denotes that the last figure has not been computed.

413. *Terms added to the Arguments and to the Constants.*

$$\delta w_1 = +5''\cdot 8 t_c^2 + \Sigma 0''\cdot 001\, C \sin\psi\,;$$

ψ	C	ψ	C
$13T - 8V + 313^\circ\cdot 9$	237	$2D - l + 21T - 20V + 273^\circ\cdot 0$	126
$Q - 4T + 239^\circ$	3	$2D - l + 8T - 12V + 303^\circ$	33
$8M - 4T + 310^\circ$	3	$2F - 2D + 6T - 5V + 270^\circ$	54
$9M - 5T + 305^\circ$	8	$3l - 2D + 24(T - V)$	10
$11M - 6T + 335^\circ$	6	$D + 12T - 15V + 262^\circ$	13
$13M - 7T + 19^\circ$	6	$D + 25T - 23V + 190^\circ$	13
$15M - 8T + 43^\circ$	26	$F + 24T - 23V + 285^\circ$	3
$17M - 9T + 63^\circ$	4	$D + l - F + 17T - 18V + 75^\circ$	8
$119^\circ\cdot 0 t_c + 152^\circ$	4*	$2D - l + 5T - 4Q + 113^\circ$	3
$20^\circ\cdot 2 t_c + 41^\circ\cdot 1$	84*	$2D - l + T - 3Q + 105^\circ$	75
$l + 3T - 10V - 2^\circ\cdot 6 t_c + 33^\circ$	31*	$2F - l + 3T - 4Q + 67^\circ$	3
$l + 16T - 18V - 1^\circ\cdot 0 t_c + 151^\circ\cdot 0$	1427*	$4D - 3l + 25M - 23T + 67^\circ$	4*
$l + 29T - 26V + 112^\circ\cdot 0$	108	$D - F + 2M + 165^\circ$	17
$l + 21(T - V)$	30	$w_3 + 276^\circ\cdot 0$	282
		$w_3 + 1^\circ\cdot 4 t_c$	7317

$$\begin{aligned}\delta w_2 = & -38''\cdot 3 t_c^2 - 0''\cdot 118 \sin(l + 16T - 18V + 151^\circ\cdot 0) \\ & + 0''\cdot 840 \sin(w_3 + 276^\circ\cdot 2) - 2''\cdot 092 \sin(w_3 + 1^\circ\cdot 4 t_c) \\ & + \text{the ten periodic terms in } \delta w_1 \text{ whose angles are independent of } w_1, w_2, w_3\,;\end{aligned}$$

$$\begin{aligned}\delta w_3 = & +6''\cdot 5 t_c^2 + 0''\cdot 172 \sin(l + 16T - 18V + 151^\circ\cdot 0) + 1''\cdot 86 \sin(w_3 + 290^\circ\cdot 1) \\ & + 15''\cdot 58 \sin(w_3 - 0^\circ\cdot 9 t_c + 276^\circ\cdot 2) + 96''\cdot 69 \sin(w_3 + 1^\circ\cdot 4 t_c)\,;\end{aligned}$$

$$\begin{aligned}\delta T = & +1''\cdot 89 \sin(13T - 8V + 134^\circ) + 0''\cdot 20 \sin(15M - 8T + 216^\circ) \\ & - 6''\cdot 40 \sin(20^\circ\cdot 2 t_c + 41^\circ) - 0''\cdot 27 \sin(119^\circ\cdot 0 t_c + 152^\circ)\,;\end{aligned}$$

$$\delta\varpi' = 0^\circ\cdot 323 t_c\,,$$

$$\delta J = +0^\circ\cdot 33 \sin(38^\circ\cdot 5 t_c + 115^\circ)\,, \quad \delta S = -0^\circ\cdot 83 \sin(38^\circ\cdot 5 t_c + 115^\circ)\,;$$

$$\begin{aligned}\delta\gamma = & -0''\cdot 083 \cos(w_3 + 290^\circ\cdot 1) - 0''\cdot 698 \cos(w_3 - 0^\circ\cdot 9 t_c + 276^\circ\cdot 2) \\ & - 4''\cdot 351 \cos(w_3 + 1^\circ\cdot 4 t_c)\,;\end{aligned}$$

$$\delta e' = -\cdot 00248\, e' t_c\,.$$

414. *True longitude* $=\Sigma 0''\cdot 001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T}-\mathrm{V}) + \alpha^\circ\}$.

$\theta = 0$

j	i	α	C
0	1	0·0	822
	2	179·8	307
	3	359·3	42
	4	0	46
	5	0	33
	6	0	24
	7	0	17
	8	0	12
	9	0	8
	10	0	6
	11	0	4
	21	0	3
1	−2	254	10
	−1	84	16
	1	82	42
	2	272·9	348
	3	271·7	176
	5	271	4
	6	272	6
	7	272	4
2	−18	209	50
	−1	27	3
	1	25	5
	2	33	3
	3	199·0	92
	4	204	26
	5	17	9
	6	207	4
3	5	114	26

$\theta = 2\mathrm{D}$

j	i	α	C
0	−11	0	3
	−10	0	5
	−9	0	6
	−8	0	8
	−7	0	8
0	−6	0	11
	−5	0	11
	−4	0	8
	−3	180	34
	−2	0	36
	−1	180	23
	1	0·0	99
	2	179·5	136
	3	178	13
	4	180	4
	18	0	3
1	1	232	3
	2	271·0	40
	3	271·5	37
	4	89	5
	20	273	3
−1	−11	78	3
	−10	78	4
	−9	78	4
	−8	78	4
	−7	78	5
	−6	78	5
	−5	84	7
	−4	78	7
	−2	271	19
	1	98	9
	2	281	13
	3	281	3
2	3	199	11
	4	202	3
	5	20	4
−2	−6	162	6
	−3	342	15
	−2	7	3
	15	151	4
	18	151	10

$\theta = l$

j	i	α	C
0	−7	180	4
	−6	180	5
	−5	180	6
	−4	180	6
	−3	180	8
	−2	0·0	61
	−1	180·0	129
	1	0·0	152
	2	180·0	48
	3	180·0	127
	4	180	11
1	−2	258	4
	1	75	8
	2	271	46
	3	272	40
	4	272	5
	5	92	4
	23	272	6
−1	−3	268	32
	−2	264	46
	−1	104	9
	1	102	3
	2	282	7
	3	280	7
2	−18	209	3
	3	210	14
	4	205	4
	5	19	3
	6	198	16
−2	−5	161	3
	−4	336	4
	−3	331	15
3	5	115	4
−3	5	65	4

True longitude $= \Sigma 0''\cdot 001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T} - \mathrm{V}) + \alpha^\circ\}$.

$\theta = 2\mathrm{D} - l$

j	i	α	C
0	−10	180	3
	−9	180	6
	−8	180	8
	−7	180	13
	−6	180	22
	−5	180	38
	−4	180·0	83
	−3	180·0	658
	−2	0·0	137
	−1	180	13
	1	0·0	133
	2	179·6	157
	3	178	14
	4	180	3
	18	0	11
1	−2	259	3
	−1	270	3
	2	271·4	65
	3	271·9	49
	4	90	5
−1	−8	269	3
	−7	269	5
	−6	269	8
	−5	269	24
	−4	89	30
	−2	268	38
	1	101	9
	2	281	13
	17	253	3
2	3	200	19
	4	202	4
	5	19	4
−2	−7	340	3
	−6	162·6	79
	−5	165	4

$\theta = 2\mathrm{D} - l$

j	i	α	C
−2	−3	340	24
	−2	6	3
	15	151	25
3	5	114	4

$\theta = 2\mathrm{D} + l$

j	i	α	C
0	−5	0	3
	−3	0	4
	−2	0	4
	1	0	11
	2	180	15
	3	180	3
1	2	271	5
	3	271	4
−1	−2	269	3

$\theta = 4\mathrm{D} - l$

j	i	α	C
0	1	180	6
	2	0	7

$\theta = 2l$

j	i	α	C
0	−2	0	5
	−1	180	10
	1	0	10
	2	180	6
	3	180	9
1	2	273	3
−1	−2	267	3

$\theta = 2l - 2\mathrm{D}$

j	i	α	C
0	−2	0	7
	−1	180	5
	2	180	3
	3	180·0	73

$\theta = 2l - 2\mathrm{D}$

j	i	α	C
0	4	0	3
1	4	92	4
−1	−3	268	3
	−2	268	3
2	6	17·4	62

$\theta = 2l - 4\mathrm{D}$

j	i	α	C
0	3	0	8

$\theta = 3l - 2\mathrm{D}$

j	i	α	C
0	3	180	3

$\theta = -4\mathrm{D}$

j	i	α	C
0	3	0	7

$\theta = -\mathrm{D}$

j	i	α	C
1	3	273	5

$\theta = l - \mathrm{D}$

j	i	α	C
1	3	273	11

$\theta = w_2$

j	i	α	C
2	3	216	19
−2	−5	255	3
	−4	255	9
	−3	75	16
	−2	75	5

$\theta = w_3 + 2\mathrm{F}$

j	i	α	C
2	3	216	4
−2	−3	40	4

$\theta = w_3 \pm l$

j	i	α	C
2	3	216	3

415. *True longitude* $=\Sigma 0''\cdot 01\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M}-\mathrm{T}) + \alpha^\circ\}$.

$\theta = 0$

j''	i	α	C
0	1	180	11
	2	180·2	195
	3	357	14
	4	349	5
1	−3	260	6
	1	224·4	327
	2	212·4	38
	3	212·5	48
	4	331	10
2	2	244·8	93
	3	245	20
	4	244	14
	5	62	6
3	3	277	16
	4	276	13
	5	275	6
	6	94	3

$\theta = 2\mathrm{D}$

j''	i	α	C
0	−2	0	5
	1	180	4
	2	181	44
	3	0	5
1	1	224	23
	2	212	6
	3	214	8
	4	37	3
−1	−5	149	3
	−2	328	3
	−1	317	23
	3	280	3
2	2	244	5
	3	244	4
	4	246	4
−2	−2	297	8

$\theta = l$

j''	i	α	C
0	−3	180	3
	−2	0	38
	−1	0	4
	1	180	5
	2	180	43
	3	0	3
	4	180	3
1	1	223·3	73
	2	212	10
	3	213	13
	5	210	9
−1	−3	330	9
	−2	327	8
	−1	306·3	74
2	2	245	17
	3	245	5
	4	244	3
	6	63	6
−2	−4	296	3
	−3	295	5
	−2	295	18
3	3	277	3
	4	276	3
−3	−4	264	3

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−5	180	3
	−4	182	20
	−3	0	5
	−2	0	13
	−1	0	3
	1	180	8
	2	181·0	61
	3	353	5

$\theta = 2\mathrm{D} - l$

j''	i	α	C
1	1	220	31
	2	212	11
	3	214	14
	4	27	3
−1	−6	149	3
	−5	151	43
	−4	329	3
	−3	327	3
	−2	328	6
	−1	320	35
	3	280	4
2	2	244	11
	3	244	6
	4	245	5
−2	−6	298	33
	−3	296	3
	−2	297	14

$\theta = 2\mathrm{D} + l$

j''	i	α	C
0	2	180	6
1	1	82	3
−1	−1	93	3

$\theta = 2l$

j''	i	α	C
0	−2	0	3
	2	180	3
1	1	232	3
−1	−1	308	3

$\theta = 2l - 2\mathrm{D}$

j''	i	α	C
0	−2	0	4
1	5	209	17
2	6	244	18

416. *True longitude* $= \Sigma 0''{\cdot}001\, C \sin \{\theta + j''\mathrm{J} + i(\mathrm{J} - \mathrm{T}) + a^\circ\}$.

$\theta = 0$

j''	i	a	C
0	1	178·8	643
	2	359·6	187
	3	7	10
1	−3	257	6
	−2	274	18
	0	289·9	87
	1	241·5	165
	2	352·0	52
	3	355	4
2	−1	250	10
	0	324	5
	1	238	25
	2	344	6
3	1	230	3

$\theta = 2\mathrm{D}$

j''	i	a	C
0	−3	0	4
	−2	180·0	70
	−1	1	33
	1	178·5	167
	2	359·2	85
	3	13	7
1	0	349	27
	1	237	35
	2	352	15
−1	−2	8	30
	−1	303	6
	0	184	33
	2	273	9
	3	102	6
2	1	236	5
	2	345	3
−2	0	200	3
	1	110	6

$\theta = l$

j''	i	a	C
0	−2	180	36
	−1	1·0	144
	1	179·0	158
	2	180·0	190
	3	21	5
1	−2	274	6
	0	282·3	62
	1	242	39
	2	352·5	96
−1	−2	188	7
	−1	298	35
	0	257·2	63
	2	273	6
	3	286	8
2	0	326	7
	1	238	5
	2	343	4
−2	−1	302	5
	0	214	7

$\theta = 2\mathrm{D} - l$

j''	i	a	C
0	−4	180	4
	−3	182	22
	−2	180·3	1137
	−1	1	51
	1	178·4	211
	2	359·2	89
	3	14	6
1	−3	261	5
	−2	310	13
	0	5·5	56
	1	237·0	46
	2	352	20
−1	−3	187	6
	−2	7·5	436

$\theta = 2\mathrm{D} - l$

j''	i	a	C
−1	−1	296	18
	0	174·2	60
	2	273	16
	3	102	7
2	1	237	6
	2	344	3
−2	−2	19	5
	−1	291	3

$\theta = 2\mathrm{D} + l$

j''	i	a	C
0	−2	180	3
	−1	1	5
	1	178	21
	2	359	7
1	0	353	4
	1	237	5
	2	352	3
−1	0	182	4
	2	273	3

$\theta = 4\mathrm{D} - l$

j''	i	a	C
0	−2	180	7
	1	358	9
	2	179	5
−1	−2	7	3

$\theta = 2l$

j''	i	a	C
0	−2	180	3
	−1	2	11
	1	178	12
	2	180	10
1	0	293	5
	1	239	3
−1	−1	301	3
	0	247	5

True longitude $= \Sigma 0''\cdot 001\, C \sin \{\theta + j''\mathrm{J} + i(\mathrm{J} - \mathrm{T}) + \alpha^\circ\}$.

$\theta = 2l - 2\mathrm{D}$

j''	i	α	C
0	−2	180	5
	−1	2	11
	1	0	3
	2	179·9	240
1	2	172·5	284
2	2	163	3

$\theta = 2l - 4\mathrm{D}$

j''	i	α	C
0	2	0	9
1	2	173	7

$\theta = 3l - 2\mathrm{D}$

j''	i	α	C
0	2	180	7
1	2	172	5

$\theta = w_3$

j''	i	α	C
1	0	45	5
2	0	168	6

417. *True longitude* $= \Sigma 0''\cdot 001\, C \sin \{\theta + j''\mathrm{S} + i(\mathrm{S} - \mathrm{T}) + \alpha^\circ\}$.

$\theta = 0$

j''	i	α	C
0	1	179·6	42
	2	0	8
1	0	273	21
	1	257	13
2	0	297	3

$\theta = 2\mathrm{D}$

j''	i	α	C
0	1	180	10
	2	0	5
1	0	270	4
	1	257	3
−1	0	255	4

$\theta = l$

j''	i	α	C
0	−1	0	6
	1	180	10
	2	180	3
1	0	263	12
	1	257	3
−1	−1	283	3
	0	277	12

$\theta = 2\mathrm{D} - l$

j''	i	α	C
0	−2	180	19
	1	180	14
	2	0	4
1	0	271	6
	1	257	3
−1	−2	271	5
	0	267	6

418. *True longitude* $= \Sigma 0''\cdot 001\, C \sin \theta$.

θ	C	θ	C
$2\mathrm{D}$	10	$2l' + 228^\circ$	4
$2\mathrm{D} - l$	39	$l + l' + 180^\circ$	6
$2\mathrm{F} - l$	4	$l - l' + 180^\circ$	6
$l' + 180^\circ$	35	$2w_3 + 2^\circ\cdot 8t_c + 180^\circ$	38

419. *Latitude* $= \Sigma 0''\cdot 001\, C \sin\{\theta + j\mathrm{T} + i(\mathrm{T}-\mathrm{V}) + a^\circ\}$.

$\theta = \pm\mathrm{F}$

j	i	a	C
0	1	0	9
	2	180	4
1	2	273	6

$\theta = \mathrm{F} + 2\mathrm{D}$

j	i	a	C
0	−2	0	3
	1	0	5
	2	180	9

$\theta = -\mathrm{F} + 2\mathrm{D}$

j	i	a	C
0	−7	180	3
	−6	180	5
	−5	180	9
	−4	180	23
	−3	0	45
	−2	0	21
	−1	180	5
	1	0	16
	2	180	23
1	2	271	9
	3	272	6
−1	−5	270·0	68
	−2	269	6
2	3	199	3
−2	−3	341	5

$\theta = \pm\mathrm{F} + l$

j	i	a	C
0	−2	0	3
	−1	180	4
	1	0	4
	2	180	3
	3	180	6

$\theta = \mathrm{F} + 2\mathrm{D} - l$

j	i	a	C
0	−4	180	4
	−3	180	29
	−2	0	6
	−1	0	5
	1	0	6
	2	180	8
1	2	271	3
−2	−6	162	4

$\theta = -\mathrm{F} + 2\mathrm{D} - l$

j	i	a	C
0	−4	180	4
	−3	180	31
	−2	0	6
	−1	0	5
	1	0	6
	2	180	8
1	2	271	3
−2	−6	162	4

$\theta = \mathrm{F} - 2\mathrm{D} + l$

j	i	a	C
1	5	90	4

$\theta = \mathrm{F} - 2\mathrm{D} - l$

j	i	a	C
1	5	270	4

$\theta = \pm\mathrm{F} + 2l - 2\mathrm{D}$

j	i	a	C
0	3	180	3
2	6	18	3

$\theta = w_1$

j	i	a	C
0	−6	285	3
	−5	285	5
	−4	285	6
	−3	285	9
	−2	285	14
	−1	285	27
	1	105	15
	2	105	6
	3	105	3
2	3	215·6	77
−2	−7	255	3
	−6	255	5
	−5	255	9
	−4	255	25
	−3	51·6	74
	−2	75	18
	−1	75	10
	1	75	6
	2	75	4
	3	75	3
3	5	125	30
−3	−5	67	7

$\theta = w_1 - 2\mathrm{D}$

j	i	a	C
2	3	36	4

$\theta = w_1 + l$

j	i	a	C
2	3	216	4
−2	−3	75	4

$\theta = w_1 - l$

j	i	a	C
2	3	36	3
−2	−3	255	4

420. *Latitude* $= \Sigma 0''\cdot 001\, C \sin\{\theta + j''\mathrm{M} + i(\mathrm{M}-\mathrm{T}) + \alpha^\circ\}$.

$\theta = \pm \mathrm{F}$

j''	i	α	C
0	2	180	3

$\theta = \mathrm{F} - 2\mathrm{D}$

j''	i	α	C
0	−2	0	8
	2	180	3
1	1	223	5
−1	−1	316	5

$\theta = \pm \mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	−2	0	3

$\theta = w_1$

j''	i	α	C
1	1	345	10

421. *Latitude* $= \Sigma 0''\cdot 001\, C \sin\{\theta + j''\mathrm{J} + i(\mathrm{J}-\mathrm{T}) + \alpha^\circ\}$.

$\theta = \pm \mathrm{F}$

j''	i	α	C
0	1	180	9
1	0	37	9

$\theta = -\mathrm{F} + 2\mathrm{D}$

j''	i	α	C
0	−3	180	5
	−2	0	26
	−1	0	6
	1	180	29
	2	0	12
1	0	350	7
	1	237	6
−1	0	181	4
	2	273	3

$\theta = \mathrm{F} + 2\mathrm{D}$

j''	i	α	C
0	−1	180	3
	1	180	8
	2	0	3

$\theta = \pm \mathrm{F} + l$

j''	i	α	C
0	−1	0	6
	1	180	8
	2	180	8
1	0	301	4
	2	353	4
−1	0	240	4

$\theta = \pm \mathrm{F} + l - 2\mathrm{D}$

j''	i	α	C
0	−2	180	4
	−1	0	11
	2	0	51
1	2	172	20

$\theta = \pm \mathrm{F} + 2l - 2\mathrm{D}$

j''	i	α	C
0	2	180	6
1	2	172	5

$\theta = w_1$

j''	i	α	C
1	0	34	7
2	0	168	35
2	0	24	18

422. *Latitude* $= \Sigma 0''\cdot 001\, C \sin \psi$.

ψ	C	ψ	C
$-\mathrm{F} + 2\mathrm{D}$	5	$w_1 + w_3 + 2^\circ\cdot 8 l_e$	83
$w_1 + 180^\circ$	17	$w_1 - w_3 - 2\mathrm{T}$	3
$w_1 - 2\mathrm{T} + 75^\circ$	8	$w_1 + w_3 + l$	5
$w_1 - 2\mathrm{D} + 180^\circ$	7	$w_1 + w_3 - l + 180^\circ$	5

423. *Parallax* $= \Sigma 0''\cdot 001\, C \cos \psi$.

ψ	C
$2\mathrm{D} - l - 3(\mathrm{T} - \mathrm{V}) + 180^\circ$	5
$2\mathrm{D} - l - 2(\mathrm{J} - \mathrm{T}) + 180^\circ$	10
$2\mathrm{D} - l - 3\mathrm{J} + 2\mathrm{T} + 7^\circ$	4

FINIS.

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